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JHEP04(2009)037

(http://iopscience.iop.org/1126-6708/2009/04/037)

Published by IOP Publishing for SISSA

Received: *March 10, 2009*
Accepted: *March 23, 2009*
Published: *April 8, 2009*

Metric 3-Lie algebras for unitary Bagger-Lambert theories

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Abstract: We prove a structure theorem for finite-dimensional indefinite-signature metric 3-Lie algebras admitting a maximally isotropic centre. This algebraic condition indicates that all the negative-norm states in the associated Bagger-Lambert theory can be consistently decoupled from the physical Hilbert space. As an immediate application of the theorem, new examples beyond index 2 are constructed. The lagrangian for the Bagger-Lambert theory based on a general physically admissible 3-Lie algebra of this kind is obtained. Following an expansion around a suitable vacuum, the precise relationship between such theories and certain more conventional maximally supersymmetric gauge theories is found. These typically involve particular combinations of $N = 8$ super Yang-Mills and massive vector supermultiplets. A dictionary between the 3-Lie algebraic data and the physical parameters in the resulting gauge theories will thereby be provided.

Keywords: AdS-CFT Correspondence, M-Theory

ArXiv ePrint: 0902.4674

doi:10.1088/1126-6708/2009/04/037

Contents

1 Introduction and summary

The fundamental ingredient in the Bagger-Lambert-Gustavsson (BLG) model [1–3], proposed as the low-energy effective field theory on a stack of coincident M2-branes, is a metric 3-Lie algebra V on which the matter fields take values. This means that V is a real vector space with a symmetric inner product $\langle -,-\rangle$ and a trilinear, alternating 3-bracket $[-,-,-] : V \times V \times V \to V$ obeying the fundamental identity [4]

$$[x,y,[z_1,z_2,z_3]] = [[x,y,z_1],z_2,z_3] + [z_1,[x,y,z_2],z_3] + [z_1,z_2,[x,y,z_3]]\,, \qquad (1.1)$$

and the metricity condition

$$\langle [x,y,z_1], z_2\rangle = -\langle z_1, [x,y,z_2]\rangle \ , \tag{1.2}$$

for all $x, y, z_i \in V$. We say that V is indecomposable if it is not isomorphic to an orthogonal direct sum of nontrivial metric 3-Lie algebras. Every indecomposable metric 3-Lie algebra gives rise to a BLG model and this motivates their classification. It is natural to attempt this classification in increasing index — the index of an inner product being the dimension of the maximum negative-definite subspace. In other words, index 0 inner products are positive-definite (called euclidean here), index 1 are lorentzian, et cetera. To this date there is a classification up to index 2, which we now review.

It was conjectured in [5] and proved in [6] (see also [7, 8]) that there exists a unique nonabelian indecomposable metric 3-Lie algebra of index 0. It is the simple 3-Lie algebra [4] S_4 with underlying vector space $\mathbb{R}^4$, orthonormal basis e_1, e_2, e_3, e_4, and 3-bracket

$$[e_i, e_j, e_k] = \sum_{\ell=1}^{4} \varepsilon_{ijk\ell} e_\ell \ , \tag{1.3}$$

where $\varepsilon = e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Nonabelian indecomposable 3-Lie algebras of index 1 were classified in [9] and are given either by

- the simple lorentzian 3-Lie algebra $S_{3,1}$ with underlying vector space $\mathbb{R}^4$, orthonormal basis e_0, e_1, e_2, e_3 with e_0 timelike, and 3-bracket

  $$[e_\mu, e_\nu, e_\rho] = \sum_{\sigma=0}^{3} \varepsilon_{\mu\nu\rho\sigma} s_\sigma e_\sigma \ , \tag{1.4}$$

  where $s_0 = -1$ and $s_i = 1$ for $i = 1, 2, 3$; or

- $W(\mathfrak{g})$, with underlying vector space $\mathfrak{g} \oplus \mathbb{R}u \oplus \mathbb{R}v$, where $\mathfrak{g}$ is a semisimple Lie algebra with a choice of positive-definite invariant inner product, extended to $W(\mathfrak{g})$ by declaring $u, v \perp \mathfrak{g}$ and $\langle u, u\rangle = \langle v, v\rangle = 0$ and $\langle u, v\rangle = 1$, and with 3-brackets

  $$[u,x,y] = [x,y] \qquad \text{and} \qquad [x,y,z] = -\langle [x,y], z\rangle\, v \ , \tag{1.5}$$

  for all $x, y, z \in \mathfrak{g}$.

The latter metric 3-Lie algebras were discovered independently in [10–12] in the context of the BLG model. The index 2 classification is presented in [13]. There we found two classes of solutions, termed Ia and IIIb. The former class is of the form $W(\mathfrak{g})$, but where $\mathfrak{g}$ is now a lorentzian semisimple Lie algebra, whereas the latter class will be recovered as a special case of the results in this paper and hence will be described in more detail below.

Let us now discuss the BLG model from a 3-algebraic perspective. The V-valued matter fields in the BLG model [1–3] comprise eight bosonic scalars X and eight fermionic Majorana spinors Ψ in three-dimensional Minkowski space $\mathbb{R}^{1,2}$. Triality allows one to take the scalars X and fermions Ψ to transform respectively in the vector and chiral spinor representations of the $\mathfrak{so}(8)$ R-symmetry. These matter fields are coupled to a nondynamical

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gauge field A which is valued in $\Lambda^2 V$ and described by a so-called twisted Chern-Simons term in the Bagger-Lambert lagrangian [1, 3]. The inner product $\langle -, - \rangle$ on V is used to describe the kinetic terms for the matter fields X and Ψ in the Bagger-Lambert lagrangian. Therefore if the index of V is positive (i.e. not euclidean signature) then the associated BLG model is not unitary as a quantum field theory, having 'wrong' signs for the kinetic terms for those matter fields in the negative-definite directions on V, thus carrying negative energy.

Indeed, for the BLG model based on the index-1 3-Lie algebra $W(\mathfrak{g})$, one encounters just this problem. Remarkably though, as noted in the pioneering works [10–12], here the matter field components X^v and Ψ^v along precisely one of the two null directions (u, v) in $W(\mathfrak{g})$ never appear in any of the interaction terms in the Bagger-Lambert lagrangian. Since the interactions are governed only by the structure constants of the 3-Lie algebra then this property simply follows from the absence of v on the left hand side of any of the 3-brackets in (1.5). Indeed the one null direction v spans the centre of $W(\mathfrak{g})$ and the linear equations of motion for the matter fields along v force the components X^u and Ψ^u in the other null direction u to take constant values (preservation of maximal supersymmetry in fact requires $\Psi^u = 0$). By expanding around this maximally supersymmetric and gauge-invariant vacuum defined by the constant expectation value of X^u, one can obtain a unitary quantum field theory. Use of this strategy in [12] gave the first indication that the resulting theory is nothing but $N = 8$ super Yang-Mills theory on $\mathbb{R}^{1,2}$ with the euclidean semi-simple gauge algebra $\mathfrak{g}$. The super Yang-Mills theory gauge coupling here being identified with the SO(8)-norm of the constant X^u. This procedure is somewhat reminiscent of the novel Higgs mechanism introduced in [14] in the context of the Bagger-Lambert theory based on the euclidean Lie 3-algebra S_4. In that case an $N = 8$ super Yang-Mills theory with $\mathfrak{su}(2)$ gauge algebra is obtained, but with an infinite set of higher order corrections suppressed by inverse powers of the gauge coupling. As found in [12], the crucial difference is that there are no such corrections present in the lorentzian case.

Of course, one must be wary of naively integrating out the free matter fields X^v and Ψ^v in this way since their absence in any interaction terms in the Bagger-Lambert lagrangian gives rise to an enhanced global symmetry that is generated by shifting them by constant values. To account for this degeneracy in the action functional, in order to correctly evaluate the partition function, one must gauge the shift symmetry and perform a BRST quantisation of the resulting theory. Fixing this gauged shift symmetry allows one to set X^v and Ψ^v equal to zero while the equations of motion for the new gauge fields sets X^u constant and $\Psi^u = 0$. Indeed this more rigorous treatment has been carried out in [15, 16] whereby the perturbative equivalence between the Bagger-Lambert theory based on $W(\mathfrak{g})$ and maximally supersymmetric Yang-Mills theory with euclidean gauge algebra $\mathfrak{g}$ was established (see also [17]). Thus the introduction of manifest unitarity in the quantum field theory has come at the expense of realising an explicit maximal superconformal symmetry in the BLG model for $W(\mathfrak{g})$, i.e. scale-invariance is broken by a nonzero vacuum expectation value for X^u. It is perhaps worth pointing out that the super Yang-Mills description seems to have not captured the intricate structure of a particular 'degenerate' branch of the classical maximally supersymmetric moduli space in the BLG model for $W(\mathfrak{g})$ found in [9]. The occurrence of this branch can be understood to arise from a degenerate limit of the theory wherein the scale $X^u = 0$ and maximal superconformal symmetry is restored. However, as found in [15, 16], the maximally superconformal unitary theory obtained by expanding around $X^u = 0$ describes a rather trivial free theory for eight scalars and fermions, whose moduli space does not describe said degenerate branch of the original moduli space.

Consider now a general indecomposable metric 3-Lie algebra with index r of the form $V = \bigoplus_{i=1}^{r}(\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus W$, where $\langle u_i, u_j\rangle = 0 = \langle v_i, v_j\rangle$, $\langle u_i, v_j\rangle = \delta_{ij}$ and W is a euclidean vector space. As explained in section 2.4 of [13], one can ensure that none of the null components X^{v_i} and Ψ^{v_i} of the matter fields appear in any of the interactions in the associated Bagger-Lambert lagrangian provided that no v_i appear on the left hand side of any of the 3-brackets on V. This guarantees one has an extra shift symmetry for each of these null components suggesting that all the associated negative-norm states in the spectrum of this theory can be consistently decoupled after gauging all the shift symmetries and following BRST quantisation of the gauged theory. A more invariant way of stating the aforementioned criterion is that V should admit a *maximally isotropic centre*: that is, a subspace $Z \subset V$ of dimension equal to the index of the inner product on V, on which the inner product vanishes identically and which is central, so that $[Z, V, V] = 0$ in the obvious notation. The null directions v_i defined above along which we require the extra shift symmetries are thus taken to provide a basis for Z. In [13] we classified all indecomposable metric 3-Lie algebras of index 2 with a maximally isotropic centre. There are nine families of such 3-Lie algebras, which were termed type IIIb in that paper. In the present paper we will prove a structure theorem for general metric 3-Lie algebras which admit a maximally isotropic centre, thus characterising them fully. Although the structure theorem falls short of a classification, we will argue that it is the best possible result for this problem. The bosonic contributions to the Bagger-Lambert lagrangians for such 3-Lie algebras will be computed but we will not perform a rigorous analysis of the physical theory in the sense of gauging the shift symmetries and BRST quantisation. We will limit ourselves to expanding the theory around a suitable maximally supersymmetric and gauge-invariant vacuum defined by a constant expectation value for X^{u_i} (with $\Psi^{u_i} = 0$). This is the obvious generalisation of the procedure used in [12] for the lorentzian theory and coincides with that used more recently in [18] for more general 3-Lie algebras. We will comment explicitly on how all the finite-dimensional examples considered in section 4 of [18] can be recovered from our formalism.

As explained in sections 2.5 and 2.6 of [13], two more algebraic conditions are necessary in order to interpret the BLG model based on a general metric 3-Lie algebra with maximally isotropic centre as an M2-brane effective field theory. Firstly, the 3-Lie algebra should admit a (nonisometric) conformal automorphism that can be used to absorb the formal coupling dependence in the BLG model. In [13] we determined that precisely four of the nine IIIb families of index 2 3-Lie algebras with maximally isotropic centre satisfy this condition. Secondly, parity invariance of the BLG model requires the 3-Lie algebra to admit an isometric antiautomorphism. This symmetry is expected of an M2-brane effective field theory based on the assumption that it should arise as an IR superconformal fixed point of $N = 8$ super Yang-Mills theory. In [13] we determined that each of the four IIIb families of index 2 3-Lie algebras admitting said conformal automorphism also admitted an isometric antiautomorphism.

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It is worth emphasising that the motivation for the two conditions above is distinct from that which led us to demand a maximally isotropic centre. The first two are required only for an M-theoretic interpretation while the latter is a basic physical consistency condition to ensure that the resulting quantum field theory is unitary. Moreover, even given a BLG model based on a 3-Lie algebra satisfying all three of these conditions, it is plain to see that the procedure we shall follow must generically break the initial conformal symmetry since it has introduced scales into the problem corresponding to the vacuum expectation values of X^{u_i}. It is inevitable that this breaking of scale-invariance will also be a feature resulting from a more rigorous treatment in terms of gauging shift symmetries and BRST quantisation.

Thus we shall concentrate just on the unitarity condition and, for the purposes of this paper, we will say that a metric 3-Lie algebra is *(physically) admissible* if it is indecomposable and admits a maximally isotropic centre. The first part of the present paper will be devoted in essence to characterising finite-dimensional admissible 3-Lie algebras. The second part will describe the general structure of the gauge theories which result from expanding the BLG model based on these physically admissible 3-Lie algebras around a given vacuum expectation value for X^{u_i}. Particular attention will be paid to explaining how the 3-Lie algebraic data translates into physical parameters of the resulting gauge theories.

This paper is organised as follows. Section 2 is concerned with the proof of theorem 2, which is outlined at the start of that section. The theorem may be paraphrased as stating that every finite-dimensional admissible 3-Lie algebra of index $r > 0$ is constructed as follows. We start with the following data:

- for each $\alpha = 1, \ldots, N$, a nonzero vector $0 \neq \kappa^\alpha \in \mathbb{R}^r$ with components κ^α_i, a positive real number $\lambda_\alpha > 0$ and a compact simple Lie algebra $\mathfrak{g}_\alpha$;
- for each $\pi = 1, \ldots, M$, a two-dimensional euclidean vector space E_π with a complex structure H_π, and two linearly independent vectors $\eta^\pi, \zeta^\pi \in \mathbb{R}^r$;
- a euclidean vector space E_0 and $K \in \Lambda^3\mathbb{R}^r \otimes E_0$ obeying the quadratic equations

$$\langle K_{ijn}, K_{k\ell m}\rangle - \langle K_{ijm}, K_{nk\ell}\rangle + \langle K_{ij\ell}, K_{mnk}\rangle - \langle K_{ijk}, K_{\ell mn}\rangle = 0,$$

  where $\langle -, -\rangle$ is the inner product on E_0;
- and $L \in \Lambda^4\mathbb{R}^r$.

On the vector space

$$V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\alpha=1}^{N} \mathfrak{g}_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0,$$

we define the following inner product extending the inner product on E_π and E_0:

- $\langle u_i, v_j\rangle = \delta_{ij}$, $\langle u_i, u_j\rangle = 0$, $\langle v_i, v_j\rangle = 0$ and u_i, v_j are orthogonal to the $\mathfrak{g}_\alpha$, E_π and E_0; and

- on each $\mathfrak{g}_\alpha$ we take $-\lambda_\alpha$ times the Killing form.

This makes V above into an inner product space of index r. On V we define the following 3-brackets, with the tacit assumption that any 3-bracket not listed here is meant to vanish:

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x_0] &= -\sum_{k=1}^{r} \langle K_{ijk}, x_0 \rangle v_k \\
[u_i, u_j, x_\pi] &= (\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi) H_\pi x_\pi \\
[u_i, x_\pi, y_\pi] &= \langle H_\pi x_\pi, y_\pi \rangle \sum_{j=1}^{r} (\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi) v_j \\
[u_i, x_\alpha, y_\alpha] &= \kappa_i^\alpha [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha \rangle \sum_{i=1}^{r} \kappa_i^\alpha v_i,
\end{aligned} \tag{1.6}$$

for all $x_0 \in E_0$, $x_\pi, y_\pi \in E_\pi$, and $x_\alpha, y_\alpha, z_\alpha \in \mathfrak{g}_\alpha$. The resulting metric 3-Lie algebra has a maximally isotropic centre spanned by the v_i. It is indecomposable provided that there is no $x_0 \in E_0$ which is perpendicular to all the K_{ijk}, whence in particular $\dim E_0 \leq \binom{r}{3}$. The only non-explicit datum in the above construction are the K_{ijk} since they are subject to certain quadratic equations. However we will see that these equations are trivially satisfied for $r < 5$. Hence the above results constitutes, in principle, a classification for indices 3 and 4, extending the classification of index 2 in [13].

Using this structure theorem we are able to calculate the lagrangian for the BLG model associated with a general physically admissible 3-Lie algebra. For the sake of clarity, we shall focus on just the bosonic contributions since the resulting theories will have a canonical maximally supersymmetric completion. Upon expanding this theory around the maximally supersymmetric vacuum defined by constant expectation values X^{u_i} (with all the other fields set to zero) we will obtain standard $N = 8$ supersymmetric (but nonconformal) gauge theories with moduli parametrised by particular combinations of the data appearing in theorem 2 and the vacuum expectation values X^{u_i}. It will be useful to think of the vacuum expectation values X^{u_i} as defining a linear map, also denoted $X^{u_i} : \mathbb{R}^r \to \mathbb{R}^8$, sending $\xi \mapsto X^\xi := \sum_{i=1}^r \xi_i X^{u_i}$. Indeed it will be found that the physical gauge theory parameters are naturally expressed in terms of components in the image of this map. That is, in general, we find that neither the data in theorem 2 nor the vacuum expectation values X^{u_i} on their own appear as physical parameters which instead arise from certain projections of the components of the data in theorem 2 onto X^{u_i} in $\mathbb{R}^8$.

The resulting Bagger-Lambert lagrangian will be found to factorise into a sum of decoupled maximally supersymmetric gauge theories on each of the euclidean components $\mathfrak{g}_\alpha$, E_π and E_0. The physical content and moduli on each component can be summarised as follows:

- On each $\mathfrak{g}_\alpha$ one has an $N = 8$ super Yang-Mills theory. The gauge symmetry is based on the simple Lie algebra $\mathfrak{g}_\alpha$. The coupling constant is given by $\|X^{\kappa^\alpha}\|$, which denotes the SO(8)-norm of the image of $\kappa^\alpha \in \mathbb{R}^r$ under the linear map X^{u_i}. The seven scalar fields take values in the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ which is orthogonal to the direction defined by X^{κ^α}. (If $X^{\kappa^\alpha} = 0$, for a given value of α, one obtains a degenerate limit corresponding to a maximally superconformal free theory for eight scalar fields and eight fermions valued in $\mathfrak{g}_\alpha$.)

- On each plane E_π one has a pair of identical free abelian $N = 8$ massive vector supermultiplets. The bosonic fields in each such supermultiplet comprise a massive vector and six massive scalars. The mass parameter is given by $\|X^{\eta^\pi} \wedge X^{\zeta^\pi}\|$, which corresponds to the area of the parallelogram in $\mathbb{R}^8$ defined by the vectors X^{η^π} and X^{ζ^π} in the image of the map X^{u_i}. The six scalar fields inhabit the $\mathbb{R}^6 \subset \mathbb{R}^8$ which is orthogonal to the plane spanned by X^{η^π} and X^{ζ^π}. (If $\|X^{\eta^\pi} \wedge X^{\zeta^\pi}\| = 0$, for a given value of π, one obtains a degenerate massless limit where the vector is dualised to a scalar, again corresponding to a maximally superconformal free theory for eight scalar fields and eight fermions valued in E_π.) Before gauge-fixing, this theory can be understood as an $N = 8$ super Yang-Mills theory with gauge symmetry based on the four-dimensional Nappi-Witten Lie algebra $\mathfrak{d}(E_\pi, \mathbb{R})$. Moreover we explain how it can be obtained from a particular truncation of an $N = 8$ super Yang-Mills theory with gauge symmetry based on any euclidean semisimple Lie algebra with rank 2, which may provide a more natural D-brane interpretation.

- On E_0 one has a decoupled $N = 8$ supersymmetric theory involving eight free scalar fields and an abelian Chern-Simons term. Since none of the matter fields are charged under the gauge field in this Chern-Simons term then its overall contribution is essentially trivial on $\mathbb{R}^{1,2}$.

Note added. During the completion of this work the paper [18] appeared whose results have noticeable overlap with those found here. In particular, they also describe the physical properties of BLG models based on certain finite-dimensional 3-Lie algebras with index greater than 1 admitting a maximally isotropic centre. The structure theorem we prove here for such 3-Lie algebras allows us to extend some of their results and make general conclusions about the nature of those unitary gauge theories which arise from BLG models based on physically admissible 3-Lie algebras. In terms of our data in theorem 2, the explicit finite-dimensional examples considered in section 4 of [18] all have $K_{ijk} = 0 = L_{ijkl}$ with only one J_{ij} nonzero. This is tantamount to taking the index $r = 2$. The example in sections 4.1 and 4.2 of [18] has $\kappa^\alpha = 0$ (i.e. no $\mathfrak{g}_\alpha$ part) while the example in section 4.3 has $\kappa^\alpha = (1, 0)^t$. These are isomorphic to two of the four physically admissible IIIb families of index 2 3-Lie algebras found in [13].

2 Towards a classification of admissible metric 3-Lie algebras

In this section we will prove a structure theorem for finite-dimensional indecomposable metric 3-Lie algebras admitting a maximally isotropic centre. We think it is of pedagogical

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value to first rederive the similar structure theorem for metric Lie algebras using a method similar to the one we will employ in the more involved case of metric 3-Lie algebras.

2.1 Metric Lie algebras with maximally isotropic centre

Recall that a Lie algebra $\mathfrak{g}$ is said to be metric, if it possesses an ad-invariant scalar product. It is said to be indecomposable if it is not isomorphic to an orthogonal direct sum of metric Lie algebras (of positive dimension). Equivalently, it is indecomposable if there are no proper ideals on which the scalar product restricts nondegenerately. A metric Lie algebra $\mathfrak{g}$ is said to have index r, if the ad-invariant scalar product has index r, which is the same as saying that the maximally negative-definite subspace of $\mathfrak{g}$ is r-dimensional. In this section we will prove a structure theorem for finite-dimensional indecomposable metric Lie algebras admitting a maximally isotropic centre, a result originally due to Kath and Olbrich [19].

2.1.1 Preliminary form of the Lie algebra

Let $\mathfrak{g}$ be a finite-dimensional indecomposable metric Lie algebra of index $r > 0$ admitting a maximally isotropic centre. Let v_i, $i = 1, \dots, r$, denote a basis for the centre. The inner product is such that $\langle v_i, v_j \rangle = 0$. Since the inner product on $\mathfrak{g}$ is nondegenerate, there exist u_i, $i = 1, \dots, r$, which obey $\langle u_i, v_j \rangle = \delta_{ij}$. It is always possible to choose the u_i such that $\langle u_i, u_j \rangle = 0$. Indeed, if the u_i do not span a maximally isotropic subspace, then redefine them by $u_i \mapsto u_i - \frac{1}{2} \sum_{j=1}^r \langle u_i, u_j \rangle v_j$ so that they do. The perpendicular complement to the $2r$-dimensional subspace spanned by the u_i and the v_j is then positive-definite. In summary, $\mathfrak{g}$ admits the following vector space decomposition

$$\mathfrak{g} = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \mathfrak{r}, \tag{2.1}$$

where $\mathfrak{r}$ is the positive-definite subspace of $\mathfrak{g}$ perpendicular to all the u_i and v_j.

Metricity then implies that the most general Lie brackets on $\mathfrak{g}$ are of the form

$$\begin{aligned} [u_i, u_j] &= K_{ij} + \sum_{k=1}^{r} L_{ijk} v_k \\ [u_i, x] &= J_i x - \sum_{j=1}^{r} \langle K_{ij}, x \rangle v_j \\ [x, y] &= [x, y]_\mathfrak{r} - \sum_{i=1}^{r} \langle x, J_i y \rangle v_i, \end{aligned} \tag{2.2}$$

where $K_{ij} = -K_{ji} \in \mathfrak{r}$, $L_{ijk} \in \mathbb{R}$ is totally skewsymmetric in the indices, $J_i \in \mathfrak{so}(\mathfrak{r})$ and $[-,-]_\mathfrak{r} : \mathfrak{r} \times \mathfrak{r} \to \mathfrak{r}$ is bilinear and skewsymmetric. Metricity and the fact that the v_i are central, means that no u_i can appear on the right-hand side of a bracket. Finally, metricity also implies that

$$\langle [x, y]_\mathfrak{r}, z \rangle = \langle x, [y, z]_\mathfrak{r} \rangle \,, \tag{2.3}$$

for all $x, y, z \in \mathfrak{r}$.

It is not hard to demonstrate that the Jacobi identity for $\mathfrak{g}$ is equivalent to the following identities on $[-,-]_{\mathfrak{r}}$, J_i and K_{ij}, whereas L_{ijk} is unconstrained:

$$[x,[y,z]_{\mathfrak{r}}]_{\mathfrak{r}} - [[x,y]_{\mathfrak{r}},z]_{\mathfrak{r}} - [y,[x,z]_{\mathfrak{r}}]_{\mathfrak{r}} = 0 \tag{2.4a}$$

$$J_i[x,y]_{\mathfrak{r}} - [J_i x,y]_{\mathfrak{r}} - [x,J_i y]_{\mathfrak{r}} = 0 \tag{2.4b}$$

$$J_i J_j x - J_j J_i x - [K_{ij},x]_{\mathfrak{r}} = 0 \tag{2.4c}$$

$$J_i K_{jk} + J_j K_{ki} + J_k K_{ij} = 0 \tag{2.4d}$$

$$\langle K_{\ell i}, K_{jk}\rangle + \langle K_{\ell j}, K_{ki}\rangle + \langle K_{\ell k}, K_{ij}\rangle = 0, \tag{2.4e}$$

for all $x,y,z \in \mathfrak{r}$.

2.1.2 $\mathfrak{r}$ is abelian

Equation (2.4a) says that $\mathfrak{r}$ is a Lie algebra under $[-,-]_{\mathfrak{r}}$, which because of equation (2.3) is metric. Being positive-definite, it is reductive, whence an orthogonal direct sum $\mathfrak{r} = \mathfrak{s} \oplus \mathfrak{a}$, where $\mathfrak{s}$ is semisimple and $\mathfrak{a}$ is abelian. We will show that for an indecomposable $\mathfrak{g}$, we are forced to take $\mathfrak{s} = 0$, by showing that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{s}^\perp$ as a metric Lie algebra.

Equation (2.4b) says that J_i is a derivation of $\mathfrak{r}$, which we know to be skewsymmetric. The Lie algebra of skewsymmetric derivations of $\mathfrak{r}$ is given by $\operatorname{ad}\mathfrak{s} \oplus \mathfrak{so}(\mathfrak{a})$. Therefore under this decomposition, we may write $J_i = \operatorname{ad} z_i + J_i^{\mathfrak{a}}$, for some unique $z_i \in \mathfrak{s}$ and $J_i^{\mathfrak{a}} \in \mathfrak{so}(\mathfrak{a})$.

Decompose $K_{ij} = K_{ij}^{\mathfrak{s}} + K_{ij}^{\mathfrak{a}}$, with $K_{ij}^{\mathfrak{s}} \in \mathfrak{s}$ and $K_{ij}^{\mathfrak{a}} \in \mathfrak{a}$. Then equation (2.4c) becomes the following two conditions

$$[z_i, z_j]_{\mathfrak{r}} = K_{ij}^{\mathfrak{s}} \tag{2.5}$$

and

$$[J_i^{\mathfrak{a}}, J_j^{\mathfrak{a}}] = 0. \tag{2.6}$$

One can now check that the $\mathfrak{s}$-component of the Jacobi identity for $\mathfrak{g}$ is automatically satisfied, whereas the $\mathfrak{a}$-component gives rise to the two equations

$$J_i^{\mathfrak{a}} K_{jk}^{\mathfrak{a}} + J_j^{\mathfrak{a}} K_{ki}^{\mathfrak{a}} + J_k^{\mathfrak{a}} K_{ij}^{\mathfrak{a}} = 0 \tag{2.7}$$

and

$$\left\langle K_{\ell i}^{\mathfrak{a}}, K_{jk}^{\mathfrak{a}}\right\rangle + \left\langle K_{\ell j}^{\mathfrak{a}}, K_{ki}^{\mathfrak{a}}\right\rangle + \left\langle K_{\ell k}^{\mathfrak{a}}, K_{ij}^{\mathfrak{a}}\right\rangle = 0. \tag{2.8}$$

We will now show that $\mathfrak{g} \cong \mathfrak{s} \oplus \mathfrak{s}^\perp$, which violates the indecomposability of $\mathfrak{g}$ unless $\mathfrak{s} = 0$. Consider the isometry φ of the vector space $\mathfrak{g}$ defined by

$$\begin{aligned}
\varphi(u_i) &= u_i - z_i - \frac{1}{2}\sum_{j=1}^{r} \langle z_i, z_j\rangle\, v_j \\
\varphi(v_i) &= v_i \\
\varphi(x) &= x + \sum_{i=1}^{r} \langle z_i, x\rangle\, v_i,
\end{aligned} \tag{2.9}$$

for all $x \in \mathfrak{r}$. Notice that if $x \in \mathfrak{a}$, then $\varphi(x) = x$. It is a simple calculation to see that for all $x, y \in \mathfrak{s}$,

$$[\varphi(u_i), \varphi(x)] = 0 \qquad \text{and} \qquad [\varphi(x), \varphi(y)] = \varphi([x, y]_{\mathfrak{r}}). \tag{2.10}$$

In other words, the image of $\mathfrak{s}$ under φ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{s}$ and commuting with its perpendicular complement in $\mathfrak{g}$. In other words, as a metric Lie algebra $\mathfrak{g} \cong \mathfrak{s} \oplus \mathfrak{s}^{\perp}$, violating the decomposability of $\mathfrak{g}$ unless $\mathfrak{s} = 0$.

In summary, we have proved the following

Lemma 1. *Let $\mathfrak{g}$ be a finite-dimensional indecomposable metric Lie algebra with index $r > 0$ and admitting a maximally isotropic centre. Then as a vector space*

$$\mathfrak{g} = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus E, \tag{2.11}$$

where E is a euclidean space, $u_i, v_i \perp E$ and $\langle u_i, v_j \rangle = \delta_{ij}$, $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 0$. Moreover the Lie bracket is given by

$$\begin{aligned}
[u_i, u_j] &= K_{ij} + \sum_{k=1}^{r} L_{ijk} v_k \\
[u_i, x] &= J_i x - \sum_{j=1}^{r} \langle K_{ij}, x \rangle \, v_j \\
[x, y] &= -\sum_{i=1}^{r} \langle x, J_i y \rangle \, v_i,
\end{aligned} \tag{2.12}$$

where $K_{ij} = -K_{ji} \in E$, $L_{ijk} \in \mathbb{R}$ is totally skewsymmetric in its indices, $J_i \in \mathfrak{so}(E)$ and in addition obey the following conditions:

$$J_i J_j - J_j J_i = 0 \tag{2.13a}$$

$$J_i K_{jk} + J_j K_{ki} + J_k K_{ij} = 0 \tag{2.13b}$$

$$\langle K_{\ell i}, K_{jk} \rangle + \langle K_{\ell j}, K_{ki} \rangle + \langle K_{\ell k}, K_{ij} \rangle = 0. \tag{2.13c}$$

The analysis of the above equations will take the rest of this section, until we arrive at the desired structure theorem.

2.1.3 Solving for the J_i

Equation (2.13a) says that the $J_i \in \mathfrak{so}(E)$ are mutually commuting, whence they span an abelian subalgebra $\mathfrak{h} \subset \mathfrak{so}(E)$. Since E is positive-definite, E decomposes as the following orthogonal direct sum as a representation of $\mathfrak{h}$:

$$E = \bigoplus_{\pi=1}^{s} E_{\pi} \oplus E_0, \tag{2.14}$$

where

$$E_0 = \{x \in E | J_i x = 0 \ \forall i\} \tag{2.15}$$

and each E_π is a two-dimensional real irreducible representation of $\mathfrak{h}$ with certain nonzero weight. Let (H_π) denote the basis for $\mathfrak{h}$ where

$$H_\pi H_\varrho = \begin{cases} 0 & \text{if } \pi \neq \varrho, \\ -\Pi_\pi & \text{if } \pi = \varrho, \end{cases} \tag{2.16}$$

where $\Pi_\pi \in \operatorname{End}(E)$ is the orthogonal projector onto E_π. Relative to this basis we can then write $J_i = \sum_\pi J_i^\pi H_\pi$, for some real numbers J_i^π.

2.1.4 Solving for the K_{ij}

Since $K_{ij} \in E$, we may decompose according to (2.14) as

$$K_{ij} = \sum_{\pi=1}^{s} K_{ij}^\pi + K_{ij}^0. \tag{2.17}$$

We may identify each E_π with a complex line where H_π acts by multiplication by i. This turns the complex number K_{ij}^π into one component of a complex bivector $K^\pi \in \Lambda^2\mathbb{C}^r$. Equation (2.13b) splits into one equation for each K^π and that equation says that

$$J_i^\pi K_{jk}^\pi + J_j^\pi K_{ki}^\pi + J_k^\pi K_{ij}^\pi = 0, \tag{2.18}$$

or equivalently that $J^\pi \wedge K^\pi = 0$, which has as unique solution $K^\pi = J^\pi \wedge t^\pi$, for some $t^\pi \in \mathbb{R}^r$. In other words,

$$K_{ij}^\pi = J_i^\pi t_j^\pi - J_j^\pi t_i^\pi. \tag{2.19}$$

Now consider the following vector space isometry $\varphi : \mathfrak{g} \to \mathfrak{g}$, defined by

$$\begin{aligned} \varphi(u_i) &= u_i - t_i - \frac{1}{2}\sum_{j=1}^{r} \langle t_i, t_j \rangle \, v_j \\ \varphi(v_i) &= v_i \\ \varphi(x) &= x + \sum_{i=1}^{r} \langle t_i, x \rangle \, v_i, \end{aligned} \tag{2.20}$$

for all $x \in E$, where $t_i \in E$ and hence $t_i = \sum_{\pi=1}^{s} t_i^\pi + t_i^0$. Under this isometry the form of the Lie algebra remains invariant, but K_{ij} changes as

$$K_{ij} \mapsto K_{ij} - J_i t_j + J_j t_i \tag{2.21}$$

and L_{ijk} changes in a manner which need not concern us here. Therefore we see that K_{ij}^π has been put to zero via this transformation, whereas K_{ij}^0 remains unchanged. In other words, we can assume without loss of generality that $K_{ij} \in E_0$, so that $J_i K_{kl} = 0$, while still being subject to the quadratic equation (2.13c).

In summary, we have proved the following theorem, originally due to Kath and Olbrich [19]:

Theorem 1. *Let $\mathfrak{g}$ be a finite-dimensional indecomposable metric Lie algebra of index $r > 0$ admitting a maximally isotropic centre. Then as a vector space*

$$\mathfrak{g} = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\pi=1}^{s} E_\pi \oplus E_0, \tag{2.22}$$

where all direct sums but the one between $\mathbb{R}u_i$ and $\mathbb{R}v_i$ are orthogonal and the inner product is as in lemma 1. Let $0 \neq J^\pi \in \mathbb{R}^r$, $K_{ij} \in E_0$ and $L_{ijk} \in \mathbb{R}$ and assume that the K_{ij} obey the following quadratic relation

$$\langle K_{\ell i}, K_{jk}\rangle + \langle K_{\ell j}, K_{ki}\rangle + \langle K_{\ell k}, K_{ij}\rangle . = 0. \tag{2.23}$$

Then the Lie bracket of $\mathfrak{g}$ is given by

$$\begin{aligned}
[u_i, u_j] &= K_{ij} + \sum_{k=1}^{r} L_{ijk} v_k \\
[u_i, x] &= J_i^\pi H_\pi x \\
[u_i, z] &= -\sum_{j=1}^{r} \langle K_{ij}, z\rangle v_j \\
[x, y] &= -\sum_{i=1}^{r} \langle x, J_i^\pi H_\pi y\rangle v_i,
\end{aligned} \tag{2.24}$$

where $x, y \in E_\pi$ and $z \in E_0$. Furthermore, indecomposability forces the K_{ij} to span all of E_0, whence $\dim E_0 \leq \binom{r}{2}$.

It should be remarked that the L_{ijk} are only defined up to the following transformation

$$L_{ijk} \mapsto L_{ijk} + \langle K_{ij}, t_k\rangle + \langle K_{ki}, t_j\rangle + \langle K_{jk}, t_i\rangle \,, \tag{2.25}$$

for some $t_i \in E_0$.

It should also be remarked that the quadratic relation (2.23) is automatically satisfied for index $r \leq 3$, whereas for index $r \geq 4$ it defines an algebraic variety. In that sense, the classification problem for indecomposable metric Lie algebras admitting a maximally isotropic centre is not tame for index $r > 3$.

2.2 Metric 3-Lie algebras with maximally isotropic centre

After the above warm-up exercise, we may now tackle the problem of interest, namely the classification of finite-dimensional indecomposable metric 3-Lie algebras with maximally isotropic centre. The proof is not dissimilar to that of theorem 1, but somewhat more involved and requires new ideas. Let us summarise the main steps in the proof.

1. In section 2.2.1 we write down the most general form of a metric 3-Lie algebra V consistent with the existence of a maximally isotropic centre Z. As a vector space,

$V = Z \oplus Z^* \oplus W$, where Z and Z^* are nondegenerately paired and W is positive-definite. Because Z is central, the 4-form $F(x,y,z,w) := \langle [x,y,z], w\rangle$ on V defines an element in $\Lambda^4(W \oplus Z)$. The decomposition

$$\Lambda^4(W \oplus Z) = \Lambda^4 W \oplus \left(\Lambda^3 W \otimes Z\right) \oplus \left(\Lambda^2 W \otimes \Lambda^2 Z\right) \oplus \left(W \otimes \Lambda^3 Z\right) \oplus \Lambda^4 Z \tag{2.26}$$

induces a decomposition of $F = \sum_{a=0}^4 F_a$, where $F_a \in \Lambda^{4-a} W \otimes \Lambda^a Z$, where the component F_4 is unconstrained.

2. The component F_0 defines the structure of a metric 3-Lie algebra on W which, if V is indecomposable, must be abelian, as shown in section 2.2.2.

3. The component F_1 defines a compatible family $[-,-]_i$ of reductive Lie algebras on W. In section 2.2.3 we show that they all are proportional to a reductive Lie algebra structure $\mathfrak{g} \oplus \mathfrak{z}$ on W, where $\mathfrak{g}$ is semisimple and $\mathfrak{z}$ is abelian.

4. In section 2.2.4 we show that the component F_2 defines a family J_{ij} of commuting endomorphisms spanning an abelian Lie subalgebra $\mathfrak{a} < \mathfrak{so}(\mathfrak{z})$. Under the action of $\mathfrak{a}$, $\mathfrak{z}$ breaks up into a direct sum of irreducible 2-planes E_π and a euclidean vector space E_0 on which the J_{ij} act trivially.

5. In section 2.2.5 we show that the component F_3 defines elements $K_{ijk} \in E_0$ which are subject to a quadratic equation.

2.2.1 Preliminary form of the 3-algebra

Let V be a finite-dimensional metric 3-Lie algebra with index $r > 0$ and admitting a maximally isotropic centre. Let v_i, $i = 1, \dots, r$, denote a basis for the centre. Since the centre is (maximally) isotropic, $\langle v_i, v_j \rangle = 0$, and since the inner product on V is nondegenerate, there exists u_i, $i = 1, \dots, r$ satisfying $\langle u_i, v_j \rangle = \delta_{ij}$. Furthermore, it is possible to choose the u_i such that $\langle u_i, u_j \rangle = 0$. The perpendicular complement W of the $2r$-dimensional subspace spanned by the u_i and v_i is therefore positive definite. In other words, V admits a vector space decomposition

$$V = \bigoplus_{i=1}^r (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus W. \tag{2.27}$$

Since the v_i are central, metricity of V implies that the u_i cannot appear in the right-hand side of any 3-bracket. The most general form for the 3-bracket for V consistent with V

being a metric 3-Lie algebra is given for all $x, y, z \in W$ by

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x] &= J_{ij} x - \sum_{k=1}^{r} \langle K_{ijk}, x \rangle v_k \\
[u_i, x, y] &= [x, y]_i - \sum_{j=1}^{r} \langle x, J_{ij} y \rangle v_j \\
[x, y, z] &= [x, y, z]_W - \sum_{i=1}^{r} \langle [x, y]_i, z \rangle v_i,
\end{aligned} \tag{2.28}$$

where $J_{ij} \in \mathfrak{so}(W)$, $K_{ijk} \in W$ and $L_{ijk\ell} \in \mathbb{R}$ are skewsymmetric in their indices, $[-,-]_i : W \times W \to W$ is an alternating bilinear map which in addition obeys

$$\langle [x, y]_i, z \rangle = \langle x, [y, z]_i \rangle , \tag{2.29}$$

and $[-,-,-]_W : W \times W \times W \to W$ is an alternating trilinear map which obeys

$$\langle [x, y, z]_W, w \rangle = - \langle [x, y, w]_W, z \rangle . \tag{2.30}$$

The following lemma is the result of a straightforward, if somewhat lengthy, calculation.

Lemma 2. *The fundamental identity (1.1) of the 3-Lie algebra V defined by (2.28) is equivalent to the following conditions, for all $t, w, x, y, z \in W$:*

$$[t, w, [x, y, z]_W]_W = [[t, w, x]_W, y, z]_W + [x, [t, w, y]_W, z]_W + [x, y, [t, w, z]_W]_W \tag{2.31a}$$

$$\begin{aligned}
[w, [x, y, z]_W]_i &= [[w, x]_i, y, z]_W + [x, [w, y]_i, z]_W + [x, y, [w, z]_i]_W & \text{(2.31b)}\\
[x, y, [z, t]_i]_W &= [z, t, [x, y]_i]_W + [[x, y, z]_W, t]_i + [z, [x, y, t]_W]_i & \text{(2.31c)}\\
J_{ij}[x, y, z]_W &= [J_{ij}x, y, z]_W + [x, J_{ij}y, z]_W + [x, y, J_{ij}z]_W & \text{(2.31d)}\\
J_{ij}[x, y, z]_W - [x, y, J_{ij}z]_W &= [[x, y]_i, z]_j - [[x, y]_j, z]_i & \text{(2.31e)}\\
[x, y, K_{ijk}]_W &= J_{jk}[x, y]_i + J_{ki}[x, y]_j + J_{ij}[x, y]_k & \text{(2.31f)}\\
[J_{ij}x, y, z]_W &= [[x, y]_i, z]_j + [[y, z]_j, x]_i + [[z, x]_i, y]_j & \text{(2.31g)}\\
J_{ij}[x, y, z]_W &= [z, [x, y]_j]_i + [x, [y, z]_j]_i + [y, [z, x]_j]_i & \text{(2.31h)}\\
[x, y, K_{ijk}]_W &= J_{ij}[x, y]_k - [J_{ij}x, y]_k - [x, J_{ij}y]_k & \text{(2.31i)}\\
J_{ik}[x, y]_j - J_{ij}[x, y]_k &= [J_{jk}x, y]_i + [x, J_{jk}y]_i & \text{(2.31j)}\\
[x, J_{jk}y]_i &= [J_{ij}x, y]_k + [J_{ki}x, y]_j + J_{jk}[x, y]_i & \text{(2.31k)}\\
[K_{ijk}, x]_\ell &= [K_{\ell ij}, x]_k + [K_{\ell jk}, x]_i + [K_{\ell ki}, x]_j & \text{(2.31l)}\\
[K_{ijk}, x]_\ell - [K_{ij\ell}, x]_k &= (J_{ij}J_{k\ell} - J_{k\ell}J_{ij})\, x & \text{(2.31m)}\\
[x, K_{jk\ell}]_i &= (J_{jk}J_{i\ell} + J_{k\ell}J_{ij} + J_{j\ell}J_{ki})\, x & \text{(2.31n)}\\
J_{im}K_{jk\ell} &= J_{ij}K_{k\ell m} + J_{ik}K_{\ell mj} + J_{i\ell}K_{jkm} & \text{(2.31o)}
\end{aligned}$$

$$J_{ij}K_{k\ell m} = J_{\ell m}K_{ijk} + J_{mk}K_{ij\ell} + J_{k\ell}K_{ijm} \tag{2.31p}$$

$$\langle K_{ijm}, K_{nk\ell}\rangle + \langle K_{ijk}, K_{\ell mn}\rangle = \langle K_{ijn}, K_{k\ell m}\rangle + \langle K_{ij\ell}, K_{mnk}\rangle\,. \tag{2.31q}$$

Of course, not all of these equations are independent, but we will not attempt to select a minimal set here, since we will be able to dispense with some of the equations easily.

2.2.2 W is abelian

Equation (2.31a) says that W becomes a 3-Lie algebra under $[-,-,-]_W$ which is metric by (2.30). Since W is positive-definite, it is reductive [6–9], whence isomorphic to an orthogonal direct sum $W = S \oplus A$, where S is semisimple and A is abelian. Furthermore, S is an orthogonal direct sum of several copies of the unique positive-definite simple 3-Lie algebra S_4 [4, 20]. We will show that as metric 3-Lie algebras $V = S \oplus S^\perp$, whence if V is indecomposable then $S = 0$ and $W = A$ is abelian as a 3-Lie algebra. This is an extension of the result in [9] by which semisimple 3-Lie algebras S factorise out of one-dimensional double extensions, and we will, in fact, follow a similar method to the one in [9] by which we perform an isometry on V which manifestly exhibits a nondegenerate ideal isomorphic to S as a 3-Lie algebra.

Consider then the isometry $\varphi : V \to V$, defined by

$$\varphi(v_i) = v_i \qquad \varphi(u_i) = u_i - s_i - \frac{1}{2}\sum_{j=1}^{r}\langle s_i, s_j\rangle\, v_j \qquad \varphi(x) = x + \sum_{i=1}^{r}\langle s_i, x\rangle\, v_i, \tag{2.32}$$

for $x \in W$ and for some $s_i \in W$. (This is obtained by extending the linear map $v_i \to v_i$ and $u_i \mapsto u_i - s_i$ to an isometry of V.) Under φ the 3-brackets (2.28) take the following form

$$\begin{aligned}
[\varphi(u_i), \varphi(u_j), \varphi(u_k)] &= \varphi(K^\varphi_{ijk}) + \sum_{\ell=1}^{r} L^\varphi_{ijk\ell} v_\ell \\
[\varphi(u_i), \varphi(u_j), \varphi(x)] &= \varphi(J^\varphi_{ij}x) - \sum_{k=1}^{r}\left\langle K^\varphi_{ijk}, x\right\rangle v_k \\
[\varphi(u_i), \varphi(x), \varphi(y)] &= \varphi([x,y]^\varphi_i) - \sum_{j=1}^{r}\left\langle x, J^\varphi_{ij}y\right\rangle v_j \\
[\varphi(x), \varphi(y), \varphi(z)] &= \varphi([x,y,z]_W) - \sum_{i=1}^{r}\langle [x,y]^\varphi_i, z\rangle\, v_i,
\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}
[x,y]^\varphi_i &= [x,y]_i + [s_i, x, y]_W \\
J^\varphi_{ij}x &= J_{ij}x + [s_i, x]_j - [s_j, x]_i + [s_i, s_j, x]_W \\
K^\varphi_{ijk} &= K_{ijk} - J_{ij}s_k - J_{jk}s_i - J_{ki}s_j + [s_i, s_j]_k + [s_j, s_k]_i + [s_k, s_i]_j - [s_i, s_j, s_k]_W \\
L^\varphi_{ijk\ell} &= L_{ijk\ell} + \langle K_{jk\ell}, s_i\rangle - \langle K_{k\ell i}, s_j\rangle + \langle K_{\ell ij}, s_k\rangle - \langle K_{ijk}, s_\ell\rangle \\
&\quad - \langle s_i, J_{k\ell}s_j\rangle - \langle s_k, J_{j\ell}s_i\rangle - \langle s_j, J_{i\ell}s_k\rangle + \langle s_\ell, J_{jk}s_i\rangle + \langle s_\ell, J_{ki}s_j\rangle + \langle s_\ell, J_{ij}s_k\rangle \\
&\quad + \langle [s_i, s_j]_\ell, s_k\rangle - \langle [s_i, s_j]_k, s_\ell\rangle - \langle [s_k, s_i]_j, s_\ell\rangle - \langle [s_j, s_k]_i, s_\ell\rangle + \langle [s_i, s_j, s_k]_W, s_\ell\rangle\,.
\end{aligned} \tag{2.34}$$

Lemma 3. *There exists $s_i \in S$ such that the following conditions are met for all $x \in S$:*

$$[x, -]_i^\varphi = 0 \qquad J_{ij}^\varphi x = 0 \qquad \left\langle K_{ijk}^\varphi, x\right\rangle = 0. \tag{2.35}$$

Assuming for a moment that this is the case, the only nonzero 3-brackets involving elements in $\varphi(S)$ are

$$[\varphi(x), \varphi(y), \varphi(z)] = \varphi([x, y, z]_W), \tag{2.36}$$

and this means that $\varphi(S)$ is a nondegenerate ideal of V, whence $V = \varphi(S) \oplus \varphi(S)^\perp$. But this violates the indecomposability of V, unless $S = 0$.

Proof of the lemma. To show the existence of the s_i, let us decompose $S = S_4^{(1)} \oplus \cdots \oplus S_4^{(m)}$ into m copies of the unique simple positive-definite 3-Lie algebra S_4. As shown in [9, section 3.2], since J_{ij} and $[x, -]_i$ define skewsymmetric derivations of W, they preserve the decomposition of W into $S \oplus A$ and that of S into its simple factors. One consequence of this fact is that $J_{ij}x \in S$ for all $x \in S$ and $[x, y]_i \in S$ for all $x, y \in S$, and similarly if we substitute S for any of its simple factors in the previous statement. Notice in addition that putting $i = j$ in equation (2.31g), $[-, -]_i$ obeys the Jacobi identity. Hence on any one of the simple factors of S — let's call it generically S_4 — the bracket $[-, -]_i$ defines the structure of a four-dimensional Lie algebra. This Lie algebra is metric by equation (2.29) and positive definite. There are (up to isomorphism) precisely two four-dimensional positive-definite metric Lie algebras: the abelian Lie algebra and $\mathfrak{so}(3) \oplus \mathbb{R}$. In either case, as shown in [9, section 3.2], there exists a unique $s_i \in S_4$ such that $[s_i, x, y]_W = [x, y]_i$ for $x, y \in S_4$. (In the former case, $s_i = 0$.) Since this is true for all simple factors, we conclude that there exists $s_i \in S$ such that $[s_i, x, y]_W = [x, y]_i$ for $x, y \in S$ and for all i.

Now equation (2.31g) says that for all $x, y, z \in S$,

$$\begin{aligned}
[J_{ij}x, y, z]_W &= [[x, y]_i, z]_j + [[y, z]_j, x]_i + [[z, x]_i, y]_j \\
&= [s_j, [s_i, x, y]_W, z]_W + [s_i, [s_j, y, z]_W, x]_W + [s_j, [s_i, z, x]_W, y]_W \\
&= [[s_i, s_j, x]_W, y, x]_W, \qquad\qquad \text{using (2.31a)}
\end{aligned}$$

which implies that $J_{ij}x - [s_i, s_j, x]_W$ centralises S, and thus is in A. However, for $x \in S$, both $J_{ij}x \in S$ and $[s_i, s_j, x]_W \in S$, so that $J_{ij}x = [s_i, s_j, x]_W$. Similarly, equation (2.31i) says that for all $x, y \in S$,

$$\begin{aligned}
[x, y, K_{ijk}]_W &= J_{ij}[x, y]_k - [J_{ij}x, y]_k - [x, J_{ij}y]_k \\
&= [s_i, s_j, [s_k, x, y]_W]_W - [s_k, [s_i, s_j, x]_W, y]_W - [s_k, x, [s_i, s_j y]_W]_W \\
&= [[s_i, s_j, s_k]_W, x, y]_W, \qquad\qquad \text{using (2.31a)}
\end{aligned}$$

which implies that $K_{ijk} - [s_i, s_j, s_k]_W$ centralises S, whence $K_{ijk} - [s_i, s_j, s_k]_W = K_{ijk}^A \in A$. Finally, using the explicit formulae for J_{ij}^φ and K_{ijk}^φ in equation (2.34), we see that for all all $x \in S$,

$$\begin{aligned}
J_{ij}^\varphi x &= J_{ij}x + [s_i, x]_j - [s_j, x]_i + [s_i, s_j, x]_W \\
&= [s_i, s_j, x]_W + [s_j, s_i, x]_W - [s_i, s_j, x]_W + [s_i, s_j, x]_W = 0
\end{aligned}$$

and

$$
\begin{aligned}
K^{\varphi}_{ijk} &= K_{ijk} - J_{ij}s_k - J_{jk}s_i - J_{ki}s_j + [s_i,s_j]_k + [s_j,s_k]_i + [s_k,s_i]_j - [s_i,s_j,s_k]_W \\
&= K^A_{ijk} + [s_i,s_j,s_k]_W - [s_i,s_j,s_k]_W - [s_j,s_k,s_i]_W - [s_k,s_i,s_j]_W \\
&\quad + [s_k,s_i,s_j]_W + [s_i,s_j,s_k]_W + [s_j,s_k,s_i]_W - [s_i,s_j,s_k]_W = K^A_{ijk},
\end{aligned}
$$

whence $\left\langle K^{\varphi}_{ijk}, x\right\rangle = 0$ for all $x \in S$. $\square$

We may summarise the above discussion as follows.

Lemma 4. *Let V be a finite-dimensional indecomposable metric 3-Lie algebra of index $r > 0$ with a maximally isotropic centre. Then as a vector space*

$$
V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus W, \tag{2.37}
$$

where W is positive-definite, $u_i, v_i \perp W$, $\langle u_i, u_j\rangle = 0$, $\langle v_i, v_j\rangle = 0$ and $\langle u_i, v_j\rangle = \delta_{ij}$. The v_i span the maximally isotropic centre. The nonzero 3-brackets are given by

$$
\begin{aligned}
[u_i,u_j,u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i,u_j,x] &= J_{ij}x - \sum_{k=1}^{r} \langle K_{ijk}, x\rangle\, v_k \\
[u_i,x,y] &= [x,y]_i - \sum_{j=1}^{r} \langle x, J_{ij}y\rangle\, v_j \\
[x,y,z] &= -\sum_{i=1}^{r} \langle [x,y]_i, z\rangle\, v_i,
\end{aligned} \tag{2.38}
$$

for all $x,y,z \in W$ and for some $L_{ijk\ell} \in \mathbb{R}$, $K_{ijk} \in W$, $J_{ij} \in \mathfrak{so}(W)$, all of which are totally skewsymmetric in their indices, and bilinear alternating brackets $[-,-]_i : W \times W \to W$ satisfying equation (2.29). Furthermore, the fundamental identity of the 3-brackets (2.38) is equivalent to the following conditions on K_{ijk}, J_{ij} and $[-,-]_i$:

$$[x,[y,z]_i]_j = [[x,y]_j,z]_i + [y,[x,z]_j]_i \tag{2.39a}$$

$$[[x,y]_i,z]_j = [[x,y]_j,z]_i \tag{2.39b}$$

$$J_{ij}[x,y]_k = [J_{ij}x,y]_k + [x,J_{ij}y]_k \tag{2.39c}$$

$$0 = J_{j\ell}[x,y]_i + J_{\ell i}[x,y]_j + J_{ij}[x,y]_\ell \tag{2.39d}$$

$$[K_{ijk},x]_\ell - [K_{ij\ell},x]_k = (J_{ij}J_{k\ell} - J_{k\ell}J_{ij})\,x \tag{2.39e}$$

$$[x,K_{jk\ell}]_i = (J_{jk}J_{i\ell} + J_{k\ell}J_{ij} + J_{j\ell}J_{ki})\,x \tag{2.39f}$$

$$J_{ij}K_{k\ell m} = J_{\ell m}K_{ijk} + J_{mk}K_{ij\ell} + J_{k\ell}K_{ijm} \tag{2.39g}$$

$$0 = \langle K_{ijn}, K_{k\ell m}\rangle + \langle K_{ij\ell}, K_{mnk}\rangle - \langle K_{ijm}, K_{nk\ell}\rangle - \langle K_{ijk}, K_{\ell mn}\rangle\,. \tag{2.39h}$$

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There are less equations in (2.39) than are obtained from (2.31) by simply making W abelian. It is not hard to show that the equations in (2.39) imply the rest. The study of equations (2.39) will take us until the end of this section. The analysis of these conditions will break naturally into several steps. In the first step we will solve equations (2.39a) and (2.39b) for the $[-,-]_i$. We will then solve equations (2.39c) and (2.39d), which will turn allow us to solve equations (2.39e) and (2.39f) for the J_{ij}. Finally we will solve equation (2.39g). We will not solve equation (2.39h). In fact, this equation defines an algebraic variety (an intersection of conics) which parametrises these 3-algebras.

2.2.3 Solving for the $[-,-]_i$

Condition (2.39a) for $i = j$ says that $[-,-]_i$ defines a Lie algebra structure on W, denoted $\mathfrak{g}_i$. By equation (2.29), $\mathfrak{g}_i$ is a metric Lie algebra. Since the inner product on W is positive-definite, $\mathfrak{g}_i$ is reductive, whence $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i] \oplus \mathfrak{z}_i$, where $\mathfrak{s}_i := [\mathfrak{g}_i, \mathfrak{g}_i]$ is the semisimple derived ideal of $\mathfrak{g}_i$ and $\mathfrak{z}_i$ is the centre of $\mathfrak{g}_i$. The following lemma will prove useful.

Lemma 5. *Let $\mathfrak{g}_i$, $i = 1, \dots, r$, be a family of reductive Lie algebras sharing the same underlying vector space W and let $[-,-]_i$ denote the Lie bracket of $\mathfrak{g}_i$. Suppose that they satisfy equations* (2.39a) *and* (2.39b) *and in addition that one of these Lie algebras, $\mathfrak{g}_1$ say, is simple. Then for all $x, y \in W$,*

$$[x, y]_i = \kappa_i [x, y]_1, \tag{2.40}$$

where $\kappa_i \in \mathbb{R}$.

Proof. Equation (2.39a) says that for all $x \in W$, $\operatorname{ad}_i x := [x, -]_i$ is a derivation of $\mathfrak{g}_j$, for all i, j. In particular, $\operatorname{ad}_1 x$ is a derivation of $\mathfrak{g}_i$. Since derivations preserve the centre, $\operatorname{ad}_1 x : \mathfrak{z}_i \to \mathfrak{z}_i$, whence the subspace $\mathfrak{z}_i$ is an ideal of $\mathfrak{g}_1$. Since by hypothesis, $\mathfrak{g}_1$ is simple, we must have that either $\mathfrak{z}_i = W$, in which case $\mathfrak{g}_i$ is abelian and the lemma holds with $\kappa_i = 0$, or else $\mathfrak{z}_i = 0$, in which case $\mathfrak{g}_i$ is semisimple. It remains therefore to study this case.

Equation (2.39a) again says that $\operatorname{ad}_i x$ is a derivation of $\mathfrak{g}_1$. Since all derivations of $\mathfrak{g}_1$ are inner, this means that there is some element y such that $\operatorname{ad}_i x = \operatorname{ad}_1 y$. This element is moreover unique because ad_1 has trivial kernel. In other words, this defines a linear map

$$\psi_i : \mathfrak{g}_i \to \mathfrak{g}_1 \qquad \text{by} \qquad \operatorname{ad}_i x = \operatorname{ad}_1 \psi_i x \qquad \forall x \in W. \tag{2.41}$$

This linear map is a vector space isomorphism since $\ker \psi_i \subset \ker \operatorname{ad}_i = 0$, for $\mathfrak{g}_i$ semisimple. Now suppose that $I \lhd \mathfrak{g}_i$ is an ideal, whence $\operatorname{ad}_i(x) I \subset I$ for all $x \in \mathfrak{g}_i$. This means that $\operatorname{ad}_1(y) I \subset I$ for all $y \in \mathfrak{g}_1$, whence I is also an ideal of $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is simple, this means that $I = 0$ or else $I = W$; in other words, $\mathfrak{g}_i$ is simple.

Now for all $x, y, z \in W$, we have

$$\begin{aligned}
[\psi_i [x, y]_i, z]_1 &= [[x, y]_i, z]_i && \text{by equation (2.41)}\\
&= [x, [y, z]_i]_i - [y, [x, z]_i]_i && \text{by the Jacobi identity of } \mathfrak{g}_i\\
&= [\psi_i x, [\psi_i y, z]_1]_1 - [\psi_i y, [\psi_i x, z]_1]_1 && \text{by equation (2.41)}\\
&= [[\psi_i x, \psi_i y]_1, z]_1 && \text{by the Jacobi identity of } \mathfrak{g}_1
\end{aligned}$$

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and since $\mathfrak{g}_1$ has trivial centre, we conclude that

$$\psi_i[x,y]_i = [\psi_i x, \psi_i y]_1,$$

whence $\psi_i : \mathfrak{g}_i \to \mathfrak{g}_1$ is a Lie algebra isomorphism.

Next, condition (2.39b) says that $\mathrm{ad}_1[x,y]_i = \mathrm{ad}_i[x,y]_1$, whence using equation (2.41), we find that $\mathrm{ad}_1[x,y]_i = \mathrm{ad}_1 \psi_i[x,y]_1$, and since ad_1 has trivial kernel, $[x,y]_i = \psi_i[x,y]_1$. We may rewrite this equation as $\mathrm{ad}_i x = \psi_i \,\mathrm{ad}_1 x$ for all x, which again by virtue of (2.41), becomes $\mathrm{ad}_1 \psi_i x = \psi_i \,\mathrm{ad}_1 x$, whence ψ_i commutes with the adjoint representation of $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is simple, Schur's lemma says that ψ_i must be a multiple, κ_i say, of the identity. In other words, $\mathrm{ad}_i x = \kappa_i \,\mathrm{ad}_1 x$, which proves the lemma. $\square$

Let us now consider the general case when none of the $\mathfrak{g}_i$ are simple. Let us focus on two reductive Lie algebras, $\mathfrak{g}_i = \mathfrak{z}_i \oplus \mathfrak{s}_i$, for $i = 1,2$ say, sharing the same underlying vector space W. We will further decompose $\mathfrak{s}_i$ into its simple ideals

$$\mathfrak{s}_i = \bigoplus_{\alpha=1}^{N_i} \mathfrak{s}_i^\alpha . \tag{2.42}$$

For every $x \in W$, $\mathrm{ad}_1 x$ is a derivation of $\mathfrak{g}_2$, whence it preserves the centre $\mathfrak{z}_2$ and each simple ideal $\mathfrak{s}_2^\beta$. This means that $\mathfrak{z}_2$ and $\mathfrak{s}_2^\beta$ are themselves ideals of $\mathfrak{g}_1$, whence

$$\mathfrak{z}_2 = E_0 \oplus \bigoplus_{\alpha\in I_0} \mathfrak{s}_1^\alpha \qquad \text{and} \qquad \mathfrak{s}_2^\beta = E_\beta \oplus \bigoplus_{\alpha\in I_\beta} \mathfrak{s}_1^\alpha \qquad \forall\beta \in \{1,2,\dots,N_2\}\,, \tag{2.43}$$

and where the index sets $I_0, I_1, \dots, I_{N_2}$ define a partition of $\{1,\dots,N_1\}$, and

$$\mathfrak{z}_1 = E_0 \oplus E_1 \oplus \dots \oplus E_{N_2} \tag{2.44}$$

is an orthogonal decomposition of $\mathfrak{z}_1$. But now notice that the restriction of $\mathfrak{g}_1$ to $E_\beta \oplus \bigoplus_{\alpha\in I_\beta} \mathfrak{s}_1^\alpha$ is reductive, whence we may apply lemma 5 to each simple $\mathfrak{s}_2^\beta$ in turn. This allows us to conclude that for each β, either $\mathfrak{s}_2^\beta = E_\beta$ or else $\mathfrak{s}_2^\beta = \mathfrak{s}_1^\alpha$, for some $\alpha \in \{1,2,\dots,N_1\}$ which depends on β, and in this latter case, $[x,y]_{\mathfrak{s}_2^\beta} = \kappa[x,y]_{\mathfrak{s}_1^\alpha}$, for some nonzero constant κ.

This means that, given any one Lie algebra $\mathfrak{g}_i$, any other Lie algebra $\mathfrak{g}_j$ in the same family is obtained by multiplying its simple factors by some constants (which may be different in each factor and may also be zero) and maybe promoting part of its centre to be semisimple.

The metric Lie algebras $\mathfrak{g}_i$ induce the following orthogonal decomposition of the underlying vector space W. We let $W_0 = \bigcap_{i=1}^r \mathfrak{z}_i$ be the intersection of all the centres of the reductive Lie algebras $\mathfrak{g}_i$. Then we have the following orthogonal direct sum $W = W_0 \oplus \bigoplus_{\alpha=1}^N W_\alpha$, where restricted to each $W_{\alpha>0}$ at least one of the Lie algebras, $\mathfrak{g}_i$ say, is simple and hence all other Lie algebras $\mathfrak{g}_{j\neq i}$ are such that for all $x, y \in W_\alpha$,

$$[x,y]_j = \kappa_{ij}^\alpha [x,y]_i \qquad \exists\kappa_{ij}^\alpha \in \mathbb{R}. \tag{2.45}$$

To simplify the notation, we define a semisimple Lie algebra structure $\mathfrak{g}$ on the perpendicular complement of W_0, whose Lie bracket $[-,-]$ is defined in such a way that for all

$x, y \in W_\alpha$, $[x, y] := [x, y]_i$, where $i \in \{1, 2, \ldots, r\}$ is the smallest such integer for which the restriction of $\mathfrak{g}_i$ to W_α is simple. (That such an integer i exists follows from the definition of W_0 and of the W_α.) It then follows that the restriction to W_α of every other $\mathfrak{g}_{j \neq i}$ is a (possibly zero) multiple of $\mathfrak{g}$.

We summarise this discussion in the following lemma, which summarises the solution of equations (2.39a) and (2.39b).

Lemma 6. *Let $\mathfrak{g}_i$, $i = 1, \ldots, r$, be a family of metric Lie algebras sharing the same underlying euclidean vector space W and let $[-,-]_i$ denote the Lie bracket of $\mathfrak{g}_i$. Suppose that they satisfy equations (2.39a) and (2.39b). Then there is an orthogonal decomposition*

$$W = W_0 \oplus \bigoplus_{\alpha=1}^{N} W_\alpha, \tag{2.46}$$

where

$$[x, y]_i = \begin{cases} 0 & \text{if } x, y \in W_0; \\ \kappa_i^\alpha [x, y] & \text{if } x, y \in W_\alpha, \end{cases} \tag{2.47}$$

for some $\kappa_i^\alpha \in \mathbb{R}$ and where $[-,-]$ are the Lie brackets of a semisimple Lie algebra $\mathfrak{g}$ with underlying vector space $\bigoplus_{\alpha=1}^{N} W_\alpha$.

2.2.4 Solving for the J_{ij}

Next we study the equations (2.39c) and (2.39d), which involve only J_{ij}. Equation (2.39c) says that each J_{ij} is a derivation over the $\mathfrak{g}_k$ for all i, j, k. Since derivations preserve the centre, every J_{ij} preserves the centre of every $\mathfrak{g}_k$ and hence it preserves their intersection W_0. Since J_{ij} preserves the inner product, it also preserves the perpendicular complement of W_0 in W, which is the underlying vector space of the semisimple Lie algebra $\mathfrak{g}$ of the previous lemma. Equation (2.39c) does not constrain the component of J_{ij} acting on W_0 since all the $[-,-]_k$ vanish there, but it does constrain the components of J_{ij} acting on $\bigoplus_{\alpha=1}^{N} W_\alpha$. Fix some α and let $x, y \in W_\alpha$. Then by virtue of equation (2.47), equation (2.39c) says that

$$\kappa_k^\alpha \left(J_{ij}[x, y] - [J_{ij}x, y] - [x, J_{ij}y] \right) = 0. \tag{2.48}$$

Since, given any α there will be at least some k for which $\kappa_k^\alpha \neq 0$, we see that J_{ij} is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, this derivation is inner, where there exists a unique $z_{ij} \in \mathfrak{g}$, such that $J_{ij}y = [z_{ij}, y]$ for all $y \in \mathfrak{g}$. Since the simple ideals of $\mathfrak{g}$ are submodules under the adjoint representation, J_{ij} preserves each of the simple ideals and hence it preserves the decomposition (2.46). Let z_{ij}^α denote the component of z_{ij} along W_α. Equation (2.39d) can now be rewritten for $x, y \in W_\alpha$ as

$$\kappa_i^\alpha [z_{j\ell}^\alpha, [x, y]] + \kappa_j^\alpha [z_{\ell i}^\alpha, [x, y]] + \kappa_\ell^\alpha [z_{ij}^\alpha, [x, y]] = 0. \tag{2.49}$$

Since $\mathfrak{g}$ has trivial centre, this is equivalent to

$$\kappa_i^\alpha z_{j\ell}^\alpha + \kappa_j^\alpha z_{\ell i}^\alpha + \kappa_\ell^\alpha z_{ij}^\alpha = 0, \tag{2.50}$$

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which can be written more suggestively as $\kappa^\alpha \wedge z^\alpha = 0$, where $\kappa^\alpha \in \mathbb{R}^r$ and $z^\alpha \in \Lambda^2\mathbb{R}^r \otimes W_\alpha$. This equation has as unique solution $z^\alpha = \kappa^\alpha \wedge s^\alpha$, for some $s^\alpha \in \mathbb{R}^r \otimes W_\alpha$, or in indices

$$z^\alpha_{ij} = \kappa^\alpha_i s^\alpha_j - \kappa^\alpha_j s^\alpha_i \qquad \exists s^\alpha_i \in W_\alpha. \tag{2.51}$$

Let $s_i = \sum_\alpha s^\alpha_i \in \mathfrak{g}$ and consider now the isometry $\varphi : V \to V$ defined by

$$\begin{aligned}
\varphi(v_i) &= v_i \\
\varphi(z) &= z \\
\varphi(u_i) &= u_i - s_i - \frac{1}{2}\sum_j \langle s_i, s_j\rangle\, v_j \\
\varphi(x) &= x + \sum_i \langle s_i, x\rangle\, v_i,
\end{aligned} \tag{2.52}$$

for all $z \in W_0$ and all $x \in \bigoplus_{\alpha=1}^N W_\alpha$. The effect of such a transformation on the 3-brackets (2.38) is an uninteresting modification of K_{ijk} and $L_{ijk\ell}$ and the more interesting disappearance of J_{ij} from the 3-brackets involving elements in W_α. Indeed, for all $x \in W_\alpha$, we have

$$\begin{aligned}
[\varphi(u_i), \varphi(u_j), \varphi(x)] &= [u_i - s_i, u_j - s_j, x] \\
&= [u_i, u_j, x] + [u_j, s_i, x] - [u_i, s_j, x] + [s_i, s_j, x] \\
&= J_{ij}x + [s_i, x]_j - [s_j, x]_i + \text{central terms} \\
&= [z^\alpha_{ij}, x] + \kappa^\alpha_j [s^\alpha_i, x] - \kappa^\alpha_i [s^\alpha_j, x] + \text{central terms} \\
&= [z^\alpha_{ij} + \kappa^\alpha_j s^\alpha_i - \kappa^\alpha_i s^\alpha_j, x] + \text{central terms} \\
&= 0 + \text{central terms},
\end{aligned}$$

where we have used equation (2.51).

This means that without loss of generality we may assume that $J_{ij}x = 0$ for all $x \in W_\alpha$ for any α. Now consider equation (2.39f) for $x \in \bigoplus_{\alpha=1}^N W_\alpha$. The right-hand side vanishes, whence $[K_{ijk}, x]_\ell = 0$. Also if $x \in W_0$, then $[K_{ijk}, x]_\ell = 0$ because x is central with respect to all $\mathfrak{g}_\ell$. Therefore we see that K_{ijk} is central with respect to all $\mathfrak{g}_\ell$, and hence $K_{ijk} \in W_0$.

In other words, we have proved the following

Lemma 7. *In the notation of lemma 6, the nonzero 3-brackets for V may be brought to the form*

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^r L_{ijk\ell} v_\ell \\
[u_i, u_j, x_0] &= J_{ij}x_0 - \sum_{k=1}^r \langle K_{ijk}, x_0\rangle\, v_k \\
[u_i, x_0, y_0] &= -\sum_{j=1}^r \langle x_0, J_{ij} y_0\rangle\, v_j \\
[u_i, x_\alpha, y_\alpha] &= \kappa^\alpha_i [x, y] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha\rangle \sum_{i=1}^r \kappa^\alpha_i v_i,
\end{aligned} \tag{2.53}$$

for all $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$*,* $x_0, y_0 \in W_0$ *and for some* $L_{ijk\ell} \in \mathbb{R}$*,* $K_{ijk} \in W_0$ *and* $J_{ij} \in \mathfrak{so}(W_0)$*, all of which are totally skewsymmetric in their indices.*

Since their left-hand sides vanish, equations (2.39e) and (2.39f) become conditions on $J_{ij} \in \mathfrak{so}(W_0)$:

$$J_{ij}J_{k\ell} - J_{k\ell}J_{ij} = 0, \tag{2.54}$$

$$J_{jk}J_{i\ell} + J_{k\ell}J_{ij} + J_{j\ell}J_{ki} = 0. \tag{2.55}$$

The first condition says that the J_{ij} commute, whence since the inner product on W_0 is positive-definite, they must belong to the same Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(W_0)$. Let H_π, for $\pi = 1, \dots, \lfloor \frac{\dim W_0}{2} \rfloor$, denote a basis for $\mathfrak{h}$, with each H_π corresponding to the generator of infinitesimal rotations in mutually orthogonal 2-planes in W_0. In particular, this means that $H_\pi H_\varrho = 0$ for $\pi \neq \varrho$ and that $H_\pi^2 = -\Pi_\pi$, with Π_π the orthogonal projector onto the 2-plane labelled by π. We write $J_{ij}^\pi \in \mathbb{R}$ for the component of J_{ij} along H_π. Fixing π we may think of J_{ij}^π as the components of $J^\pi \in \Lambda^2\mathbb{R}^r$. Using the relations obeyed by the H_π, equation (2.55) separates into $\lfloor \frac{\dim W_0}{2} \rfloor$ equations, one for each value of π, which in terms of J^π can be written simply as $J^\pi \wedge J^\pi = 0$. This is a special case of a Plücker relation and says that J^π is decomposable; that is, $J^\pi = \eta^\pi \wedge \zeta^\pi$ for some $\eta^\pi, \zeta^\pi \in \mathbb{R}^r$. In other words, the solution of equations (2.54) and (2.55) is

$$J_{ij} = \sum_\pi \left(\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi\right) H_\pi \tag{2.56}$$

living in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(W_0)$.

2.2.5 Solving for the K_{ijk}

It remains to solve equations (2.39g) and (2.39h) for K_{ijk}. We shall concentrate on the linear equation (2.39g). This is a linear equation on $K \in \Lambda^3\mathbb{R}^r \otimes W_0$ and says that it is in the kernel of a linear map

$$\Lambda^3\mathbb{R}^r \otimes W_0 \longrightarrow \Lambda^2\mathbb{R}^r \otimes \Lambda^3\mathbb{R}^r \otimes W_0 \tag{2.57}$$

defined by

$$K_{ijk} \mapsto J_{ij}K_{k\ell m} - J_{\ell m}K_{ijk} - J_{mk}K_{ij\ell} - J_{k\ell}K_{ijm}. \tag{2.58}$$

The expression in the right-hand side is manifestly skewsymmetric in ij and $k\ell m$ separately, whence it belongs to $\Lambda^2\mathbb{R}^r \otimes \Lambda^3\mathbb{R}^r \otimes W_0$ as stated above. For generic r (here $r \geq 5$) we may decompose

$$\Lambda^2\mathbb{R}^r \otimes \Lambda^3\mathbb{R}^r = Y^{\yng(2,2,1)}\mathbb{R}^r \oplus Y^{\yng(2,1,1,1)}\mathbb{R}^r \oplus \Lambda^5\mathbb{R}^r, \tag{2.59}$$

where $Y^{\text{Young tableau}}$ denotes the corresponding Young symmetriser representation. Then one can see that the right-hand side of (2.58) has no component in the first of the above summands and hence lives in the remaining two summands, which are isomorphic to $\mathbb{R}^r \otimes \Lambda^4\mathbb{R}^r$.

We now observe that via an isometry of V of the form

$$\begin{aligned}
\varphi(v_i) &= v_i \\
\varphi(x_\alpha) &= x_\alpha \\
\varphi(u_i) &= u_i + t_i - \frac{1}{2}\sum_j \langle t_i, t_j\rangle v_j \\
\varphi(x_0) &= x_0 - \sum_i \langle x_0, t_i\rangle v_i,
\end{aligned} \tag{2.60}$$

for $t_i \in W_0$, the form of the 3-brackets (2.53) remains invariant, but with K_{ijk} and $L_{ijk\ell}$ transforming by

$$K_{ijk} \mapsto K_{ijk} + J_{ij}t_k + J_{jk}t_i + J_{ki}t_j, \tag{2.61}$$

and

$$\begin{aligned}
L_{ijk\ell} \mapsto & L_{ijk\ell} + \langle K_{ijk}, t_\ell\rangle - \langle K_{\ell ij}, t_k\rangle + \langle K_{k\ell i}, t_j\rangle - \langle K_{jk\ell}, t_i\rangle \\
& + \langle J_{ij}t_k, t_\ell\rangle + \langle J_{ki}t_j, t_\ell\rangle + \langle J_{jk}t_i, t_\ell\rangle + \langle J_{i\ell}t_j, t_k\rangle + \langle J_{j\ell}t_k, t_i\rangle + \langle J_{k\ell}t_i, t_j\rangle\,,
\end{aligned} \tag{2.62}$$

respectively. In particular, this means that there is an ambiguity in K_{ijk}, which can be thought of as shifting it by the image of the linear map

$$\mathbb{R}^r \otimes W_0 \longrightarrow \Lambda^3\mathbb{R}^r \otimes W_0 \tag{2.63}$$

defined by

$$t_i \mapsto J_{ij}t_k + J_{jk}t_i + J_{ki}t_j. \tag{2.64}$$

The two maps (2.57) and (2.63) fit together in a complex

$$\mathbb{R}^r \otimes W_0 \longrightarrow \Lambda^3\mathbb{R}^r \otimes W_0 \longrightarrow \mathbb{R}^r \otimes \Lambda^4\mathbb{R}^r \otimes W_0, \tag{2.65}$$

where the composition vanishes *precisely* by virtue of equations (2.54) and (2.55). We will show that this complex is acyclic away from the kernel of J, which will mean that without loss of generality we can take K_{ijk} in the kernel of J subject to the final quadratic equation (2.39h).

Let us decompose W_0 into an orthogonal direct sum

$$W_0 = \begin{cases} \displaystyle\bigoplus_{\pi=1}^{(\dim W_0)/2} E_\pi, & \text{if } \dim W_0 \text{ is even, and} \\ \displaystyle\mathbb{R}w \oplus \bigoplus_{\pi=1}^{(\dim W_0 - 1)/2} E_\pi, & \text{if } \dim W_0 \text{ is odd,} \end{cases} \tag{2.66}$$

where E_π are mutually orthogonal 2-planes and, in the second case, w is a vector perpendicular to all of them. On E_π the Cartan generator H_π acts as a complex structure, and hence we may identify each E_π with a complex one-dimensional vector space and H_π with multiplication by i. This decomposition of W_π allows us to decompose $K_{ijk} = K^w_{ijk} + \sum_\pi K^\pi_{ijk}$, where the first term is there only in the odd-dimensional situation and the K^π_{ijk} are complex numbers. The complex (2.65) breaks up into $\lfloor\frac{\dim W_0}{2}\rfloor$ complexes, one for each value of π. If $J^\pi = 0$ then K^π_{ijk} is not constrained there, but if $J^\pi = \eta^\pi \wedge \zeta^\pi \neq 0$ the complex turns out to have no homology, as we now show.

Without loss of generality we may choose the vectors η^π and ζ^π to be the elementary vectors e_1 and e_2 in $\mathbb{R}^r$, so that J^π has a $J^\pi_{12} = 1$ and all other $J^\pi_{ij} = 0$. Take $i = 1$ and $j = 2$ in the cocycle condition (2.57), to obtain

$$K^\pi_{k\ell m} = J^\pi_{\ell m} K^\pi_{12k} + J^\pi_{mk} K^\pi_{12\ell} + J^\pi_{k\ell} K^\pi_{12m}. \tag{2.67}$$

It follows that if any two of $k, \ell, m > 2$, then $K^\pi_{k\ell m} = 0$. In particular $K^\pi_{1ij} = K^\pi_{2ij} = 0$ for all $i, j > 2$, whence only K^π_{12k} for $k > 2$ can be nonzero. However for $k > 2$, $K^\pi_{12k} = J^\pi_{12} e_k$, with e_k the kth elementary vector in $\mathbb{R}^r$, and hence K^π_{12k} is in the image of the map (2.63); that is, a coboundary. This shows that we may assume without loss of generality that $K^\pi_{ijk} = 0$. In summary, the only components of K_{ijk} which survive are those in the kernel of all the J_{ij}. It is therefore convenient to split W_0 into an orthogonal direct sum

$$W_0 = E_0 \oplus \bigoplus_\pi E_\pi, \tag{2.68}$$

where on each 2-plane E_π, $J^\pi = \eta^\pi \wedge \zeta^\pi \neq 0$, whereas $J_{ij} x = 0$ for all $x \in E_0$. Then we can take $K_{ijk} \in E_0$.

Finally it remains to study the quadratic equation (2.39h). First of all we mention that this equation is automatically satisfied for $r \leq 4$. To see this notice that the equation is skewsymmetric in k, ℓ, m, n, whence if $r < 4$ it is automatically zero. When $r = 4$, we have to take k, ℓ, m, n all different and hence the equation becomes

$$\langle K_{ij1}, K_{234}\rangle - \langle K_{ij2}, K_{341}\rangle + \langle K_{ij3}, K_{412}\rangle - \langle K_{ij4}, K_{123}\rangle = 0,$$

which is skewsymmetric in i, j. There are six possible choices for i, j but by symmetry any choice is equal to any other up to relabeling, so without loss of generality let us take $i = 1$ and $j = 2$, whence the first two terms are identically zero and the two remaining terms satisfy

$$\langle K_{123}, K_{412}\rangle - \langle K_{124}, K_{123}\rangle = 0,$$

which is identically true. This means that the cases of index 3 and 4 are classifiable using our results. By contrast, the case of index 5 and above seems not to be tame. An example should suffice. So let us take the case of $r = 5$ and $\dim E_0 = 1$, so that the K_{ijk} can be taken to be real numbers. The solutions to (2.39h) now describe the intersection of five quadrics in $\mathbb{R}^{10}$:

$$\begin{aligned}
K_{125}K_{134} - K_{124}K_{135} + K_{123}K_{145} &= 0\\
K_{125}K_{234} - K_{124}K_{235} + K_{123}K_{245} &= 0\\
K_{135}K_{234} - K_{134}K_{235} + K_{123}K_{345} &= 0\\
K_{145}K_{234} - K_{134}K_{245} + K_{124}K_{345} &= 0\\
K_{145}K_{235} - K_{135}K_{245} + K_{125}K_{345} &= 0,
\end{aligned}$$

whence the solutions define an algebraic variety. One possible branch is given by setting $K_{1ij} = 0$ for all i, j, which leaves undetermined K_{234}, K_{235}, K_{245} and K_{345}. There are other branches which are linearly related to this one: for instance, setting $K_{2ij} = 0$, et cetera, but there are also other branches which are not linearly related to it.

2.2.6 Summary and conclusions

Let us summarise the above results in terms of the following structure theorem.

Theorem 2. *Let V be a finite-dimensional indecomposable metric 3-Lie algebra of index $r > 0$ with a maximally isotropic centre. Then V admits a vector space decomposition into $r + M + N + 1$ orthogonal subspaces*

$$V = \bigoplus_{i=1}^{r} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\alpha=1}^{N} W_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0, \tag{2.69}$$

where W_α, E_π and E_0 are positive-definite subspaces with the E_π being two-dimensional, and where $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$ and $\langle u_i, v_j\rangle = \delta_{ij}$. The 3-Lie algebra is defined in terms of the following data:

- $0 \neq \eta^\pi \wedge \zeta^\pi \in \Lambda^2\mathbb{R}^r$ *for each* $\pi = 1, \ldots, M$,
- $0 \neq \kappa^\alpha \in \mathbb{R}^r$ *for each* $\alpha = 1, \ldots, N$,
- *a metric simple Lie algebra structure* $\mathfrak{g}_\alpha$ *on each* W_α,
- $L \in \Lambda^4\mathbb{R}^r$, *and*
- $K \in \Lambda^3\mathbb{R}^r \otimes E_0$ *subject to the equation*

$$\langle K_{ijn}, K_{k\ell m}\rangle + \langle K_{ij\ell}, K_{mnk}\rangle - \langle K_{ijm}, K_{nk\ell}\rangle - \langle K_{ijk}, K_{\ell mn}\rangle = 0,$$

by the following 3-brackets,[1]

$$\begin{aligned}
[u_i, u_j, u_k] &= K_{ijk} + \sum_{\ell=1}^{r} L_{ijk\ell} v_\ell \\
[u_i, u_j, x_0] &= -\sum_{k=1}^{r} \langle K_{ijk}, x_0\rangle v_k \\
[u_i, u_j, x_\pi] &= J^\pi_{ij} H_\pi x_\pi \\
[u_i, x_\pi, y_\pi] &= -\sum_{j=1}^{r} \left\langle x_\pi, J^\pi_{ij} H_\pi y_\pi\right\rangle v_j \\
[u_i, x_\alpha, y_\alpha] &= \kappa^\alpha_i [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha\rangle \sum_{i=1}^{r} \kappa^\alpha_i v_i,
\end{aligned} \tag{2.70}$$

[1]We understand tacitly that if a 3-bracket is not listed here it vanishes. Also every summation is written explicitly, so the summation convention is not in force. In particular, there is no sum over π in the third and fourth brackets.

for all $x_0 \in E_0$, $x_\pi, y_\pi \in E_\pi$ and $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$, and where $J^\pi_{ij} = \eta^\pi_i \zeta^\pi_j - \eta^\pi_j \zeta^\pi_i$ and H_π a complex structure on each 2-plane E_π. The resulting 3-Lie algebra is indecomposable provided that there is no $x_0 \in E_0$ which is perpendicular to all the K_{ijk}, whence in particular $\dim E_0 \leq \binom{r}{3}$.

2.3 Examples for low index

Let us now show how to recover the known classifications in index ≤ 2 from theorem 2.

Let us consider the case of minimal positive index $r = 1$. In that case, the indices i, j, k, l in theorem 2 can only take the value 1 and therefore J_{ij}, K_{ijk} and L_{ijkl} are not present. Indecomposability of V forces $E_0 = 0$ and $E_\pi = 0$, whence letting $u = u_1$ and $v = v_1$, we have $V = \mathbb{R}u \oplus \mathbb{R}v \oplus \bigoplus_{\alpha=1}^N W_\alpha$ as a vector space, with $\langle u, u\rangle = \langle v, v\rangle = 0$, $\langle u, v\rangle = 1$ and $\bigoplus_{\alpha=1}^N W_\alpha$ euclidean. The 3-brackets are:

$$
\begin{aligned}
[u, x_\alpha, y_\alpha] &= [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha\rangle\, v,
\end{aligned}
\tag{2.71}
$$

for all $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$ and where we have redefined $\kappa^\alpha [x_\alpha, y_\alpha] \to [x_\alpha, y_\alpha]$, which is a simple Lie algebra on each W_α. This agrees with the classification of lorentzian 3-Lie algebras in [9] which was reviewed in the introduction.

Let us now consider $r = 2$. According to theorem 2, those with a maximally isotropic centre may now have a nonvanishing J_{12} while K_{ijk} and L_{ijkl} are still absent. Indecomposability of V forces $E_0 = 0$. Therefore $W_0 = \bigoplus_{\pi=1}^M E_\pi$ and, as a vector space, $V = \mathbb{R}u_1 \oplus \mathbb{R}v_1 \oplus \mathbb{R}u_2 \oplus \mathbb{R}v_2 \oplus W_0 \oplus \bigoplus_{\alpha=1}^N W_\alpha$ with $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$, $\langle u_i, v_j\rangle = \delta_{ij}$, $\forall i, j = 1, 2$ and $W_0 \oplus \bigoplus_{\alpha=1}^N W_\alpha$ is euclidean. The 3-brackets are now:

$$
\begin{aligned}
[u_1, u_2, x_\pi] &= J x_\pi \\
[u_1, x_\pi, y_\pi] &= -\langle x_\pi, J y_\pi\rangle\, v_2 \\
[u_2, x_\pi, y_\pi] &= \langle x_\pi, J y_\pi\rangle\, v_1 \\
[u_1, x_\alpha, y_\alpha] &= \kappa^\alpha_1 [x_\alpha, y_\alpha] \\
[u_2, x_\alpha, y_\alpha] &= \kappa^\alpha_2 [x_\alpha, y_\alpha] \\
[x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha\rangle\, \kappa^\alpha_1 v_1 - \langle [x_\alpha, y_\alpha], z_\alpha\rangle\, \kappa^\alpha_2 v_2,
\end{aligned}
\tag{2.72}
$$

for all $x_\pi, y_\pi \in E_\pi$ and $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$. This agrees with the classification in [13] of finite-dimensional indecomposable 3-Lie algebras of index 2 whose centre contains a maximally isotropic plane. In that paper such algebras were denoted $V_{\mathrm{IIIb}}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$ with underlying vector space $\mathbb{R}(u, v) \oplus \mathbb{R}(\boldsymbol{e}_+, \boldsymbol{e}_-) \oplus E \oplus \mathfrak{l} \oplus \mathfrak{h} \oplus \mathfrak{g}$ with $\langle u, u\rangle = \langle v, v\rangle = \langle \boldsymbol{e}_\pm, \boldsymbol{e}_\pm\rangle = 0$, $\langle u, v\rangle = 1 = \langle \boldsymbol{e}_+, \boldsymbol{e}_-\rangle$ and all $\oplus$ orthogonal. The nonzero Lie 3-brackets are given by

$$
\begin{aligned}
[u, \boldsymbol{e}_-, x] &= Jx & [u, g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} \\
[u, x, y] &= \langle Jx, y\rangle\, \boldsymbol{e}_+ & [\boldsymbol{e}_-, g_1, g_2] &= [g_1, g_2]_{\mathfrak{g}} \\
[\boldsymbol{e}_-, x, y] &= -\langle Jx, y\rangle\, v & [g_1, g_2, g_3] &= -\langle [g_1, g_2]_{\mathfrak{g}}, g_3\rangle\, \boldsymbol{e}_+ - \langle [\psi g_1, g_2]_{\mathfrak{g}}, g_3\rangle\, v \\
[\boldsymbol{e}_-, h_1, h_2] &= [h_1, h_2]_{\mathfrak{h}} & [u, \ell_1, \ell_2] &= [\ell_1, \ell_2]_{\mathfrak{l}} \\
[h_1, h_2, h_3] &= -\langle [h_1, h_2]_{\mathfrak{h}}, h_3\rangle\, \boldsymbol{e}_+ & [\ell_1, \ell_2, \ell_3] &= -\langle [\ell_1, \ell_2]_{\mathfrak{l}}, \ell_3\rangle\, v,
\end{aligned}
\tag{2.73}
$$

where $x, y \in E$, $h, h_i \in \mathfrak{h}$, $g_i \in \mathfrak{g}$ and $\ell_i \in \mathfrak{l}$.

To see that this family of 3-algebras is of the type (2.72) it is enough to identify

$$u_1 \leftrightarrow u \qquad v_1 \leftrightarrow v \qquad u_2 \leftrightarrow e_- \qquad v_2 \leftrightarrow e_+ \tag{2.74}$$

as well as

$$W_0 \leftrightarrow E \qquad \text{and} \qquad \bigoplus_{\alpha=1}^{N} W_\alpha \leftrightarrow \mathfrak{l} \oplus \mathfrak{h} \oplus \mathfrak{g}, \tag{2.75}$$

where the last identification is not only as vector spaces but also as Lie algebras, and set

$$\begin{aligned} \kappa_1|_{\mathfrak{h}} &= 0 & \kappa_2|_{\mathfrak{h}} &= 1 \\ \kappa_1|_{\mathfrak{l}} &= 1 & \kappa_2|_{\mathfrak{l}} &= 0 \\ \kappa_1|_{\mathfrak{g}_\alpha} &= \psi_\alpha & \kappa_2|_{\mathfrak{g}_\alpha} &= 1, \end{aligned} \tag{2.76}$$

to obtain the map between the two families. As shown in [13] there are 9 different types of such 3-Lie algebras, depending on which of the four ingredients (E, J), $\mathfrak{l}$, $\mathfrak{h}$ or $(\mathfrak{g}, \psi)$ are present.

The next case is that of index $r = 3$, where there are up to 3 nonvanishing J_{ij} and one $K_{123} := K$, while L_{ijkl} is still not present. Indecomposability of V forces $\dim E_0 \leq 1$. As a vector space, V splits up as

$$V = \bigoplus_{i=1}^{3} (\mathbb{R}u_i \oplus \mathbb{R}v_i) \oplus \bigoplus_{\alpha=1}^{N} W_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0, \tag{2.77}$$

where all $\oplus$ are orthogonal except the second one, W_α, E_0 and E_π are positive-definite subspaces with $\dim E_0 \leq 1$, E_π being two-dimensional, and where $\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0$ and $\langle u_i, v_j\rangle = \delta_{ij}$. The 3-brackets are given by

$$\begin{aligned} [u_1, u_2, u_3] &= K \\ [u_i, u_j, x_0] &= -\sum_{k=1}^{r} \langle K_{ijk}, x_0\rangle\, v_k \\ [u_i, u_j, x_\pi] &= J_{ij}^\pi H_\pi x_\pi \\ [u_i, x_\pi, y_\pi] &= -\sum_{j=1}^{r} \left\langle x_\pi, J_{ij}^\pi H_\pi y_\pi\right\rangle v_j \\ [u_i, x_\alpha, y_\alpha] &= \kappa_i^\alpha [x_\alpha, y_\alpha] \\ [x_\alpha, y_\alpha, z_\alpha] &= -\langle [x_\alpha, y_\alpha], z_\alpha\rangle \sum_{i=1}^{r} \kappa_i^\alpha v_i, \end{aligned} \tag{2.78}$$

for all $x_0 \in E_0$, $x_\pi, y_\pi \in E_\pi$ and $x_\alpha, y_\alpha, z_\alpha \in W_\alpha$, and where $J_{ij}^\pi = \eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi$ and H_π a complex structure on each 2-plane E_π.

Finally, let us remark that the family of admissible 3-Lie algebras found in [18] are included in theorem 2. In that paper, a family of solutions to equations (2.31) was found by setting each of the Lie algebra structures $[-,-]_i$ to be nonzero in orthogonal subspaces of W. This corresponds, in the language of this paper, to the particular case of allowing precisely one κ_i^α to be nonvanishing in each W_α.

Notice that, as shown in (2.76), already in [13] there are examples of admissible 3-Lie algebras of index 2 which are not of this form as both κ_1 and κ_2 might be nonvanishing in the $\mathfrak{g}_\alpha$ factors.

To solve the rest of the equations, two ansätze are proposed in [18]:

- the trivial solution with nonvanishing J, i.e. $\kappa_i^\alpha = 0$, $K_{ijk} = 0$ for all $i, j, k = 1, \dots, r$ and for all α; and
- precisely one $\kappa_i^\alpha = 1$ for each α (and include those W_α's where all κ's are zero in W_0) and one $J_{ij} := J \neq 0$ assumed to be an outer derivation of the reference Lie algebra defined on W.

As pointed out in that paper, L_{ijkl} is not constrained by the fundamental identity, so it can in principle take any value, whereas the ansatz provided for K_{ijk} is given in terms of solutions of an equation equivalent to (2.39h). In the lagrangians considered, both L_{ijkl} and K_{ijk} are set to zero.

One thing to notice is that in all these theories there is certain redundancy concerning the index of the 3-Lie algebra. If the indices in the nonvanishing structures κ_i^α, J_{ij}, K_{ijk} and L_{ijkl} involve only numbers from 1 to r_0, then any 3-Lie algebra with such nonvanishing structures and index $r \geq r_0$ gives rise to the equivalent theories.

In this light, in the first ansatz considered, one can always define the non vanishing J to be J_{12} and then the corresponding theory will be equivalent to one associated to the index-2 3-Lie algebras considered in [13].

In the second case, the fact that J is an outer derivation implies that it must live on the abelian part of W as a Lie algebra, since the semisimple part does not possess outer derivations. This coincides with what was shown above, i.e., that $J|_{W_\alpha} = 0$ for each α. Notice that each Lie algebra $[-,-]_i$ identically vanishes in W_0, therefore the structure constants of the 3-Lie algebra do not mix J and $[-,-]_i$. The theories in [18] corresponding to this ansatz also have $K_{ijk} = 0$, whence again they are equivalent to the theory corresponding to the index-2 3-Lie algebra which was denoted $V(E, J, \mathfrak{h})$ in [13].

3 Bagger-Lambert lagrangians

In this section we will consider the physical properties of the Bagger-Lambert theory based on the most general kind of admissible metric 3-Lie algebra, as described in theorem 2.

In particular we will investigate the structure of the expansion of the corresponding Bagger-Lambert lagrangians around a vacuum wherein the scalars in half of the null directions of the 3-Lie algebra take the constant values implied by the equations of motion for the scalars in the remaining null directions, spanning the maximally isotropic centre. This technique was also used in [18] and is somewhat reminiscent of the novel Higgs mechanism that was first introduced by Mukhi and Papageorgakis [14] in the context of the Bagger-Lambert theory based on the unique simple euclidean 3-Lie algebra S_4. Recall that precisely this strategy has already been employed in lorentzian signature in [12], for the class of Bagger-Lambert theories found in [10–12] based on the unique admissible lorentzian metric 3-Lie algebra $W(\mathfrak{g})$, where it was first appreciated that this theory is perturbatively equivalent to $N = 8$ super Yang-Mills theory on $\mathbb{R}^{1,2}$ with the euclidean semisimple gauge algebra $\mathfrak{g}$. That is, there are no higher order corrections to the super Yang-Mills lagrangian here, in contrast with the infinite set of corrections (suppressed by inverse powers of the gauge coupling) found for the super Yang-Mills theory with $\mathfrak{su}(2)$ gauge algebra arising from higgsing the Bagger-Lambert theory based on S_4 in [14]. This perturbative equivalence between the Bagger-Lambert theory based on $W(\mathfrak{g})$ and maximally supersymmetric Yang-Mills theory with euclidean gauge algebra $\mathfrak{g}$ has since been shown more rigorously in [15–17].

We will show that there exists a similar relation with $N = 8$ super Yang-Mills theory after expanding around the aforementioned maximally supersymmetric vacuum the Bagger-Lambert theories based on the more general physically admissible metric 3-Lie algebras we have considered. However, the gauge symmetry in the super Yang-Mills theory is generally based on a particular indefinite signature metric Lie algebra here that will be identified in terms of the data appearing in theorem 2. The physical properties of the these Bagger-Lambert theories will be shown to describe particular combinations of decoupled super Yang-Mills multiplets with euclidean gauge algebras and free maximally supersymmetric massive vector multiplets. We will identify precisely how the physical moduli relate to the algebraic data in theorem 2. We will also note how the theories resulting from those finite-dimensional indefinite signature 3-Lie algebras considered in [18] are recovered.

3.1 Review of two gauge theories in indefinite signature

Before utilising the structural results of the previous section, let us briefly review some general properties of the maximal $N = 8$ supersymmetric Bagger-Lambert and Yang-Mills theories in three-dimensional Minkowski space that will be of interest to us, when the fields are valued in a vector space V equipped with a metric of indefinite signature. We shall denote this inner product by $\langle -, - \rangle$ and take it to have general indefinite signature $(r, r+n)$. We can then define a null basis $e_A = (u_i, v_i, e_a)$ for V, with $i = 1, \ldots, r$, $a = 1, \ldots, n$, such that $\langle u_i, v_j \rangle = \delta_{ij}$, $\langle u_i, u_j \rangle = 0 = \langle v_i, v_j \rangle$ and $\langle e_a, e_b \rangle = \delta_{ab}$.

For the sake of clarity in the forthcoming analysis, we will ignore the fermions in these theories. Needless to say that they both have a canonical maximally supersymmetric completion and none of the manipulations we will perform break any of the supersymmetries of the theories.

3.1.1 Bagger-Lambert theory

Let us begin by reviewing some details of the bosonic field content of the Bagger-Lambert theory based on the 3-bracket $[-, -, -]$ defining a metric 3-Lie algebra structure on V. The components of the canonical 4-form for the metric 3-Lie algebra are $F_{ABCD} := \langle [e_A, e_B, e_C], e_D \rangle$ (indices will be lowered and raised using the metric $\langle e_A, e_B \rangle$ and its inverse). The bosonic fields in the Bagger-Lambert theory have components X_I^A and $(\tilde{A}_\mu)^A{}_B = F^A{}_{BCD} A_\mu^{CD}$, corresponding respectively to the scalars ($I = 1, \ldots, 8$ in the vector of the $\mathfrak{so}(8)$ R-symmetry) and the gauge field ($\mu = 0, 1, 2$ on $\mathbb{R}^{1,2}$ Minkowski space). Although the supersymmetry transformations and equations of motion can be expressed in terms of $(\tilde{A}_\mu)^A_B$, the lagrangian requires it to be expressed as above in terms of A^{AB}_μ.

The bosonic part of the Bagger-Lambert lagrangian is given by

$$\mathscr{L} = -\frac{1}{2}\langle D_\mu X_I, D^\mu X_I\rangle + \mathscr{V}(X) + \mathscr{L}_{\mathrm{CS}}\,, \tag{3.1}$$

where the scalar potential is

$$\mathscr{V}(X) = -\tfrac{1}{12}\langle [X_I, X_J, X_K], [X_I, X_J, X_K]\rangle\,, \tag{3.2}$$

the Chern-Simons term is

$$\mathscr{L}_{\mathrm{CS}} = \frac{1}{2}\left(A^{AB} \wedge d\tilde{A}_{AB} + \tfrac{2}{3}A^{AB} \wedge \tilde{A}_{AC} \wedge \tilde{A}^C{}_B\right)\,, \tag{3.3}$$

and $D_\mu \phi^A = \partial_\mu \phi^A + (\tilde{A}_\mu)^A{}_B \phi^B$ defines the action on any field ϕ valued in V of the derivative D that is gauge-covariant with respect to $\tilde{A}^A{}_B$. The infinitesimal gauge transformations take the form $\delta\phi^A = -\tilde{\Lambda}^A{}_B \phi^B$ and $\delta(\tilde{A}_\mu)^A{}_B = \partial_\mu \tilde{\Lambda}^A{}_B + (\tilde{A}_\mu)^A{}_C \tilde{\Lambda}^C{}_B - \tilde{\Lambda}^A{}_C (\tilde{A}_\mu)^C{}_B$, where $\tilde{\Lambda}^A{}_B = F^A{}_{BCD}\Lambda^{CD}$ in terms of an arbitrary skewsymmetric parameter $\Lambda^{AB} = -\Lambda^{BA}$.

If we now assume that the indefinite signature metric 3-Lie algebra above admits a maximally isotropic centre which we can take to be spanned by the basis elements v_i then the 4-form components $F_{v_i ABC}$ must all vanish identically. There are two important physical consequences of this assumption. The first is that the covariant derivative $D_\mu X^{u_i}_I = \partial_\mu X^{u_i}_I$. The second is that the tensors F_{ABCD} and $F_{ABC}{}^G F_{DEFG} = F_{ABC}{}^g F_{DEFg}$ which govern all the interactions in the Bagger-Lambert lagrangian contain no legs in the v_i directions. Therefore the components $A^{v_i A}_\mu$ of the gauge field do not appear at all in the lagrangian while $X^{v_i}_I$ appear only in the free kinetic term $-D_\mu X^{u_i}_I \partial^\mu X^{v_i}_I = -\partial_\mu X^{u_i}_I \partial^\mu X^{v_i}_I$. Thus $X^{v_i}_I$ can be integrated out imposing that each $X^{u_i}_I$ be a harmonic function on $\mathbb{R}^{1,2}$ which must be a constant if the solution is to be nonsingular. (We will assume this to be the case henceforth but singular monopole-type solutions may also be worthy of investigation, as in [21].) It is perhaps just worth noting that, in addition to setting $X^{u_i}_I$ constant, one must also set the fermions in all the u_i directions to zero which is necessary and sufficient for the preservation of maximal supersymmetry here.

The upshot is that we now have $-\frac{1}{2}\langle D_\mu X_I, D^\mu X_I\rangle = -\frac{1}{2} D_\mu X^a_I D^\mu X^a_I$ (with contraction over only the euclidean directions of V) and each $X^{u_i}_I$ is taken to be constant in (3.1). Since both $X^{v_i}_I$ and $A^{v_i A}_\mu$ are now absent, it will be more economical to define $X^i_I := X^{u_i}_I$ and $A^{ia}_\mu := A^{u_i a}_\mu$ henceforth.

3.1.2 Super Yang-Mills theory

Let us now perform an analogous review for $N = 8$ super Yang-Mills theory, with gauge symmetry based on the Lie bracket $[-, -]$ defining a metric Lie algebra structure $\mathfrak{g}$ on V. The components of the canonical 3-form on $\mathfrak{g}$ are $f_{ABC} := \langle [e_A, e_B], e_C\rangle$. The bosonic fields in the theory consist of a gauge field A^A_μ and seven scalar fields X^A_I (where now $I = 1, \ldots, 7$ in the vector of the $\mathfrak{so}(7)$ R-symmetry) with all fields taking values in V. The field strength for the gauge field takes the canonical form $F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ in terms of the gauge-covariant derivative $D_\mu = \partial_\mu + [A_\mu, -]$. This theory is not scale-invariant and has a dimensionful coupling constant κ.

The bosonic part of the super Yang-Mills lagrangian is given by

$$\mathscr{L}^{\mathrm{SYM}}(A^A, X_I^A, \kappa|\mathfrak{g}) = -\frac{1}{2}\langle D_\mu X_I, D^\mu X_I\rangle - \frac{\kappa^2}{4}\langle [X_I, X_J], [X_I, X_J]\rangle - \frac{1}{4\kappa^2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle\,. \quad (3.4)$$

Noting explicitly the dependence on the data on the left hand side will be useful when we come to consider super Yang-Mills theories with a much more elaborate gauge structure.

Assuming now that $\mathfrak{g}$ admits a maximally isotropic centre, again spanned by the basis elements v_i, then the 3-form components $f_{v_i AB}$ must all vanish identically. This property implies $DX_I^{u_i} = dX_I^{u_i}$, $F^{u_i} = dA^{u_i}$ and that the tensors f_{ABC} and $f_{AB}{}^E f_{CDE} = f_{AB}{}^e f_{CDe}$ which govern all the interactions contain no legs in the v_i directions. Therefore $X_I^{v_i}$ and A^{v_i} only appear linearly in their respective free kinetic terms, allowing them to be integrated out imposing that $X_I^{u_i}$ is constant and A^{u_i} is exact. Setting the fermions in all the u_i directions to zero again ensures the preservation of maximal supersymmetry.

The resulting structure is that all the inner products using $\langle e_A, e_B\rangle$ in (3.4) are to be replaced with $\langle e_a, e_b\rangle$ while all $X_I^{u_i}$ are to be taken constant and $A^{u_i} = d\phi^{u_i}$, for some functions ϕ^{u_i}. With both $X_I^{v_i}$ and A^{v_i} now absent, it will be convenient to define $X_I^i := X_I^{u_i}$ and $\phi^i := \phi^{u_i}$ henceforth.

Let us close this review by looking in a bit more detail at the physical properties of a particular example of a super Yang-Mills theory in indefinite signature with maximally isotropic centre, whose relevance will become clear in the forthcoming sections. Four-dimensional Yang-Mills theories based on such gauge groups were studied in [22]. The gauge structure of interest is based on the lorentzian metric Lie algebra defined by the double extension $\mathfrak{d}(E, \mathbb{R})$ of an even-dimensional vector space E with euclidean inner product. Writing $V = \mathbb{R}u \oplus \mathbb{R}v \oplus E$ as a lorentzian vector space, the nonvanishing Lie brackets of $\mathfrak{d}(E, \mathbb{R})$ are given by

$$[u, x] = Jx\,, \qquad [x, y] = -\langle x, Jy\rangle\, v\,, \quad (3.5)$$

for all $x, y \in E$ where the skewsymmetric endomorphism $J \in \mathfrak{so}(E)$ is part of the data defining the double extension. The canonical 3-form for $\mathfrak{d}(E, \mathbb{R})$ therefore has only the components $f_{uab} = J_{ab}$ with respect to the euclidean basis e_a on E. It will be convenient to take J to be nondegenerate and so the eigenvalues of J^2 will be negative-definite.

We shall define the positive number $\mu^2 := X_I^u X_I^u$ as the SO(7)-norm-squared of the constant 7-vector X_I^u and the projection operator $P_{IJ}^u := \delta_{IJ} - \mu^{-2} X_I^u X_J^u$ onto the hyperplane $\mathbb{R}^6 \subset \mathbb{R}^7$ orthogonal to X_I^u. It will also be convenient to define $x^a := X_I^u X_I^a$ as the projection of the seventh super Yang-Mills scalar field along X_I^u and $\mathcal{D}\Phi := d\Phi - d\phi^u \wedge J\Phi$ where Φ can be any p-form on $\mathbb{R}^{1,2}$ taking values in E. In terms of this data, the super Yang-Mills lagrangian $\mathscr{L}^{\mathrm{SYM}}((d\phi^u, A^a), (X_I^u, X_I^a), \kappa|\mathfrak{d}(E, \mathbb{R}))$ can be more succinctly expressed as

$$-\frac{1}{2}P_{IJ}^u \mathcal{D}_\mu X_I^a \mathcal{D}^\mu X_J^a + \frac{\kappa^2\mu^2}{2}(J^2)_{ab} P_{IJ}^u X_I^a X_J^b - \frac{1}{4\kappa^2}(2\,\mathcal{D}_{[\mu}A_{\nu]}^a)(2\,\mathcal{D}^{[\mu}A^{\nu]\,a})$$
$$- \frac{1}{2\mu^2}\left(\mathcal{D}_\mu x^a + \mu^2 J^{ab} A_\mu^b\right)\left(\mathcal{D}^\mu x^a + \mu^2 J^{ac} A^{\mu\, c}\right)\,. \quad (3.6)$$

From the first line we see that the six scalar fields $P^u_{IJ}X^a_J$ are massive with mass-squared given by the eigenvalues of the matrix $-\kappa^2\mu^2(J^2)_{ab}$. All the fields couple to $d\phi^u$ through the covariant derivative $\mathcal{D}$, but the second line shows that only the seventh scalar x^a couples to the gauge field A^a. However, the gauge symmetry of (3.6) under the transformations $\delta A^a = \mathcal{D}\lambda^a$ and $\delta x^a = -\mu^2 J^{ab}\lambda^b$, for any parameter $\lambda^a \in E$, shows that x^a is in fact pure gauge and can be removed in (3.6) by fixing $\lambda^a = \mu^{-2}(J^{-1})^{ab}x^b$. The remaining gauge symmetry of (3.6) is generated by the transformations $\delta\phi^u = \alpha$ and $\delta\Phi = \alpha\, J\Phi$ for all fields $\Phi \in E$, where α is an arbitrary scalar parameter. This is obvious since $\mathcal{D} = \exp(\phi^u J) d \exp(-\phi^u J)$ and therefore, one can take $\mathcal{D} = d$ in (3.6) by fixing $\alpha = -\phi^u$.

Thus, in the gauge defined above, the lagrangian $\mathscr{L}^{\rm SYM}((d\phi^u, A^a), (X^u_I, X^a_I), \kappa|\mathfrak{d}(E,\mathbb{R}))$ becomes simply

$$-\frac{1}{2}P^u_{IJ}\partial_\mu X^a_I\partial^\mu X^a_J + \frac{\kappa^2\mu^2}{2}(J^2)_{ab}P^u_{IJ}X^a_I X^b_J - \frac{1}{4\kappa^2}(2\partial_{[\mu}A^a_{\nu]})(2\partial^{[\mu}A^{\nu]a}) + \frac{\mu^2}{2}(J^2)_{ab}A^a_\mu A^{\mu b}, \quad (3.7)$$

describing $\dim E$ decoupled free abelian $N=8$ supersymmetric massive vector multiplets, each of which contains bosonic fields given by the respective gauge field $\frac{1}{\kappa}A^a_\mu$ plus six scalars $P^u_{IJ}X^a_I$, all with the same mass-squared equal to the respective eigenvalue of $-\kappa^2\mu^2(J^2)_{ab}$.

It is worth pointing out that one can also obtain precisely the theory above from a particular truncation of an $N=8$ super Yang-Mills theory with euclidean semisimple Lie algebra $\mathfrak{g}$. If one introduces a projection operator P_{IJ} onto a hyperplane $\mathbb{R}^6 \subset \mathbb{R}^7$ then one can rewrite the seven scalar fields in this euclidean theory in terms of the six projected fields $P_{IJ}X^a_J$ living on the hyperplane plus the single scalar y^a in the complementary direction. Unlike in the lorentzian theory above however, this seventh scalar is not pure gauge. Indeed, if we expand the super Yang-Mills lagrangian (3.4) for this euclidean theory around a vacuum where y^a is constant then this constant appears as a physical modulus of the effective field theory, namely it gives rise to mass terms for the gauge field A^a and the six projected scalars $P_{IJ}X^a_J$. If one then truncates the effective field theory to the Coulomb branch, such that the dynamical fields A and $P_{IJ}X_J$ take values in a Cartan subalgebra $\mathfrak{t} < \mathfrak{g}$ (while the constant vacuum expectation value $y \in \mathfrak{g}$), then the lagrangian takes precisely the form (3.7) after making the following identifications. First one must take $E = \mathfrak{t}$ whereby the gauge field A^a and coupling κ are the the same for both theories. Second one must identify the six-dimensional hyperplanes occupied by the scalars X^a_I in both theories such that P^u_{IJ} in (3.7) is identified with P_{IJ} here. Finally, the mass matrix for the euclidean theory is $-\kappa^2[(\mathrm{ad}_y)^2]_{ab}$ which must be identified with $-\kappa^2\mu^2(J^2)_{ab}$ in (3.7). This last identification requires some words of explanation. We have defined $\mathrm{ad}_y\,\Phi := [y,\Phi]$ for all $\Phi \in \mathfrak{g}$, where $[-,-]$ denotes the Lie bracket on $\mathfrak{g}$. Since we have truncated the dynamical fields to the Cartan subalgebra $\mathfrak{t}$, only the corresponding legs of $(\mathrm{ad}_y)^2$ contribute to the mass matrix. However, clearly y must not also be contained in $\mathfrak{t}$ or else the resulting mass matrix would vanish identically. Indeed, without loss of generality, one can take y to live in the orthogonal complement $\mathfrak{t}^\perp \subset \mathfrak{g}$ since it is only these components which contribute to the mass matrix. Thus, although $(\mathrm{ad}_y)^2$ can be nonvanishing on $\mathfrak{t}$, ad_y cannot. Thus we cannot go further and equate ad_y with μJ, even though their squares agree on $\mathfrak{t}$. To summarise all this more succinctly, after the aforementioned gauge-fixing of the lorentzian theory and truncation of the euclidean theory, we have shown that

$$\mathscr{L}^{\mathrm{SYM}}\left((d\phi^u, A|_E), (X_I^u, P_{IJ}^u X_J|_E, x|_E), \kappa|\mathfrak{d}(E,\mathbb{R})\right) = \mathscr{L}^{\mathrm{SYM}}\left(A|_E, (P_{IJ}X_J|_E, y|_{E^\perp}), \kappa|\mathfrak{g}\right)\,, \tag{3.8}$$

where $E = \mathfrak{t}$, $y \in \mathfrak{t}^\perp \subset \mathfrak{g}$ is constant and $(\mathrm{ad}_y)^2 = \mu^2 J^2$ on $\mathfrak{t}$. Of course, it is not obvious that one can always solve this last equation for y in terms of a given μ and J nor indeed whether this restricts ones choice of $\mathfrak{g}$. However, it is the particular case of $\dim E = 2$ that will be of interest to us in the context of the Bagger-Lambert theory in 3.2.2 where we shall describe a nontrivial solution for any rank-2 semisimple Lie algebra $\mathfrak{g}$. Obvious generalisations of this solution give strong evidence that the equation can in fact always be solved.

3.2 Bagger-Lambert theory for admissible metric 3-Lie algebras

We will now substitute the data appearing in theorem 2 into the bosonic part of the Bagger-Lambert lagrangian (3.1), that is after having integrated out $X_I^{v_i}$ to set all $X_I^i := X_I^{u_i}$ constant.

Since we will be dealing with components of the various tensors appearing in theorem 2, we need to introduce some index notation for components of the euclidean subspace $\bigoplus_{\alpha=1}^N W_\alpha \oplus \bigoplus_{\pi=1}^M E_\pi \oplus E_0$. To this end we partition the basis $e_a = (e_{a_\alpha}, e_{a_\pi}, e_{a_0})$ on the euclidean part of the algebra, where subscripts denote a basis for the respective euclidean subspaces. For example, $a_\alpha = 1, \ldots, \dim W_\alpha$ whose range can thus be different for each α. Similarly $a_0 = 1, \ldots, \dim E_0$, while $a_\pi = 1, 2$ for each two-dimensional space E_π. Since the decomposition $\bigoplus_{\alpha=1}^N W_\alpha \oplus \bigoplus_{\pi=1}^M E_\pi \oplus E_0$ is orthogonal with respect to the euclidean metric $\langle e_a, e_b\rangle = \delta_{ab}$, we can take only the components $\langle e_{a_\alpha}, e_{b_\alpha}\rangle = \delta_{a_\alpha b_\alpha}$, $\langle e_{a_\pi}, e_{b_\pi}\rangle = \delta_{a_\pi b_\pi}$ and $\langle e_{a_0}, e_{b_0}\rangle = \delta_{a_0 b_0}$ to be nonvanishing. Since these are all just unit metrics on the various euclidean factors then we will not need to be careful about raising and lowering repeated indices, which are to be contracted over the index range of a fixed value of α, π or 0. Summations of the labels α and π will be made explicit.

In terms of this notation, we may write the data from theorem 2 in terms of the following nonvanishing components of the canonical 4-form F_{ABCD} of the algebra

$$\begin{aligned} F_{u_i a_\alpha b_\alpha c_\alpha} &= \kappa_i^\alpha f_{a_\alpha b_\alpha c_\alpha} \\ F_{u_i u_j a_\pi b_\pi} &= \left(\eta_i^\pi \zeta_j^\pi - \eta_j^\pi \zeta_i^\pi\right) \epsilon_{a_\pi b_\pi} \\ F_{u_i u_j u_k a_0} &= K_{ijka_0} \\ F_{u_i u_j u_k u_l} &= L_{ijkl}\,, \end{aligned} \tag{3.9}$$

where $f_{a_\alpha b_\alpha c_\alpha}$ denotes the canonical 3-form for the simple metric Lie algebra structure $\mathfrak{g}_\alpha$ on W_α and we have used the fact that the 2x2 matrix H_π has only components $\epsilon_{a_\pi b_\pi} = -\epsilon_{b_\pi a_\pi}$, with $\epsilon_{12} = -1$, on each 2-plane E_π.

A final point of notational convenience will be to define $Y^{AB} := X_I^A X_I^B$ and the projection $X_I^\xi := \xi_i X_I^i$ for any $\xi \in \mathbb{R}^r$. Combining these definitions allows us to write certain projections which often appear in the lagrangian like $Y^{\xi\varsigma} := X_I^\xi X_I^\varsigma$ and $Y^{\xi a} := X_I^\xi X_I^a$ for any $\xi, \varsigma \in \mathbb{R}^r$. It will sometimes be useful to write $Y^{\xi\xi} \equiv \|X^\xi\|^2 \geq 0$ where $\|X^\xi\|$ denotes the SO(8)-norm of the vector X_I^ξ. A similar shorthand will be adopted for projections of components of the gauge field, so that $A_\mu^{\xi\varsigma} := \xi_i \varsigma_j A_\mu^{ij}$ and $A_\mu^{\xi a} := \xi_i A_\mu^{ia}$.

It will be useful to note that the euclidean components of the covariant derivative $D_\mu X_I^A = \partial_\mu X_I^A + (\tilde{A}_\mu)^A{}_B X_I^B$ from section 3.1.1 can be written

$$\begin{aligned}
D_\mu X_I^{a_\alpha} &= \partial_\mu X_I^{a_\alpha} - \kappa_i^\alpha f^{a_\alpha b_\alpha c_\alpha} \left(2\, A_\mu^{i b_\alpha} X_I^{c_\alpha} + A_\mu^{b_\alpha c_\alpha} X_I^i \right) \\
&=: \mathscr{D}_\mu X_I^{a_\alpha} - 2\, B_\mu^{a_\alpha} X_I^{\kappa^\alpha} \\
D_\mu X_I^{a_\pi} &= \partial_\mu X_I^{a_\pi} + 2\, \eta_i^\pi \zeta_j^\pi \epsilon^{a_\pi b_\pi} \left(A_\mu^{ij} X_I^{b_\pi} - A_\mu^{i b_\pi} X_I^j + A_\mu^{j b_\pi} X_I^i \right) \\
&= \partial_\mu X_I^{a_\pi} + 2\, \epsilon^{a_\pi b_\pi} \left(A_\mu^{\eta^\pi \zeta^\pi} X_I^{b_\pi} - A_\mu^{\eta^\pi b_\pi} X_I^{\zeta^\pi} + A_\mu^{\zeta^\pi b_\pi} X_I^{\eta^\pi} \right) \\
D_\mu X_I^{a_0} &= \partial_\mu X_I^{a_0} - K_{ijk}{}^{a_0} A_\mu^{ij} X_I^k \ .
\end{aligned} \tag{3.10}$$

The second line defines two new quantities on each W_α, namely $B_\mu^{a_\alpha} := \frac{1}{2} f^{a_\alpha b_\alpha c_\alpha} A_\mu^{b_\alpha c_\alpha}$ and the covariant derivative $\mathscr{D}_\mu X_I^{a_\alpha} := \partial_\mu X_I^{a_\alpha} - 2\, f^{a_\alpha b_\alpha c_\alpha} \kappa_i^\alpha A_\mu^{i b_\alpha} X_I^{c_\alpha}$. The latter object is just the canonical covariant derivative with respect to the projected gauge field $\mathscr{A}_\mu^{a_\alpha} := -2\, A_\mu^{\kappa^\alpha a_\alpha}$ on each W_α. The associated field strength $\mathscr{F}_{\mu\nu} = [\mathscr{D}_\mu, \mathscr{D}_\nu]$ has components

$$\mathscr{F}^{a_\alpha} = -2\, \kappa_i^\alpha \left(dA^{i a_\alpha} - \kappa_j^\alpha f^{a_\alpha b_\alpha c_\alpha} A^{i b_\alpha} \wedge A^{j c_\alpha} \right) \ . \tag{3.11}$$

Although somewhat involved, the nomenclature above will help us understand more clearly the structure of the Bagger-Lambert lagrangian. Let us consider now the contributions to (3.1) coming from the scalar kinetic terms, the sextic potential and the Chern-Simons term in turn.

The kinetic terms for the scalar fields give

$$-\frac{1}{2} \langle D_\mu X_I, D^\mu X_I \rangle = -\frac{1}{2} \sum_{\alpha=1}^{N} D_\mu X_I^{a_\alpha} D^\mu X_I^{a_\alpha} - \frac{1}{2} \sum_{\pi=1}^{M} D_\mu X_I^{a_\pi} D^\mu X_I^{a_\pi} - \frac{1}{2} D_\mu X_I^{a_0} D^\mu X_I^{a_0} \tag{3.12}$$

which expands to

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \left\{ -\frac{1}{2} \mathscr{D}_\mu X_I^{a_\alpha} \mathscr{D}^\mu X_I^{a_\alpha} + 2\, X_I^{\kappa^\alpha} B_\mu^{a_\alpha} \mathscr{D}^\mu X_I^{a_\alpha} - 2\, Y^{\kappa^\alpha \kappa^\alpha} B_\mu^{a_\alpha} B^{\mu\, a_\alpha} \right\} \\
&\qquad + \sum_{\pi=1}^{M} \left\{ -\frac{1}{2} \partial_\mu X_I^{a_\pi} \partial^\mu X_I^{a_\pi} - 2\, \partial^\mu X_I^{a_\pi} \epsilon^{a_\pi b_\pi} \left(A_\mu^{\eta^\pi \zeta^\pi} X_I^{b_\pi} - A_\mu^{\eta^\pi b_\pi} X_I^{\zeta^\pi} + A_\mu^{\zeta^\pi b_\pi} X_I^{\eta^\pi} \right) \right. \\
&-2 \left. \left(A_\mu^{\eta^\pi \zeta^\pi} X_I^{a_\pi} - A_\mu^{\eta^\pi a_\pi} X_I^{\zeta^\pi} + A_\mu^{\zeta^\pi a_\pi} X_I^{\eta^\pi} \right) \left(A^{\mu\, \eta^\pi \zeta^\pi} X_I^{a_\pi} - A^{\mu\, \eta^\pi a_\pi} X_I^{\zeta^\pi} + A^{\mu\, \zeta^\pi a_\pi} X_I^{\eta^\pi} \right) \right\} \\
&\qquad\qquad - \frac{1}{2} \partial_\mu X_I^{a_0} \partial^\mu X_I^{a_0} + K_{ijk}{}^{a_0} A_\mu^{ij} \partial^\mu Y^{k a_0} - \frac{1}{2} K_{ijka_0} K_{lmna_0} Y^{kl} A_\mu^{ij} A^{\mu\, mn} \ .
\end{aligned} \tag{3.13}$$

The scalar potential can be written $\mathscr{V}(X) = \mathscr{V}^W(X) + \mathscr{V}^E(X) + \mathscr{V}^{E_0}(X)$ where

$$\begin{aligned}
\mathscr{V}^W(X) &= -\frac{1}{4}\sum_{\alpha=1}^{N} f^{a_\alpha b_\alpha e_\alpha} f^{c_\alpha d_\alpha e_\alpha} \left(Y^{\kappa^\alpha \kappa^\alpha} Y^{a_\alpha c_\alpha} - Y^{\kappa^\alpha a_\alpha} Y^{\kappa^\alpha c_\alpha}\right) Y^{b_\alpha d_\alpha} \\
\mathscr{V}^E(X) &= -\frac{1}{2}\sum_{\pi=1}^{M} \Big\{ Y^{a_\pi a_\pi}\left(Y^{\eta^\pi \eta^\pi} Y^{\zeta^\pi \zeta^\pi} - (Y^{\eta^\pi \zeta^\pi})^2\right) + 2\, Y^{\eta^\pi a_\pi} Y^{\zeta^\pi a_\pi} Y^{\eta^\pi \zeta^\pi} \\
&\qquad - Y^{\eta^\pi a_\pi} Y^{\eta^\pi a_\pi} Y^{\zeta^\pi \zeta^\pi} - Y^{\zeta^\pi a_\pi} Y^{\zeta^\pi a_\pi} Y^{\eta^\pi \eta^\pi} \Big\} \\
\mathscr{V}^{E_0}(X) &= -\frac{1}{12} K_{ijka_0} K_{lmna_0} Y^{il} Y^{jm} Y^{kn} \ .
\end{aligned} \tag{3.14}$$

Notice that $\mathscr{V}^{E_0}(X)$ is constant and will be ignored henceforth.

And finally, the Chern-Simons term can be written $\mathscr{L}_{\rm CS} = \mathscr{L}_{\rm CS}^W + \mathscr{L}_{\rm CS}^E + \mathscr{L}_{\rm CS}^{E_0}$ where

$$\begin{aligned}
\mathscr{L}_{\rm CS}^W &= -2\sum_{\alpha=1}^{N} B^{a_\alpha} \wedge \mathscr{F}^{a_\alpha} \\
\mathscr{L}_{\rm CS}^E &= -4\sum_{\pi=1}^{M} \left\{ \epsilon^{a_\pi b_\pi} A^{\eta^\pi a_\pi} \wedge A^{\zeta^\pi b_\pi} + 2\, A^{\eta^\pi \zeta^\pi} \wedge A^{\eta^\pi a_\pi} \wedge A^{\zeta^\pi a_\pi} - \frac{1}{2}\epsilon^{a_\pi b_\pi} A^{a_\pi b_\pi} \wedge dA^{\eta^\pi \zeta^\pi} \right\} \\
\mathscr{L}_{\rm CS}^{E_0} &= 2\, K_{ijka_0} A^{ij} \wedge dA^{ka_0} - \frac{1}{3} K_{ikla_0} K_{jmna_0} A^{ij} \wedge A^{kl} \wedge A^{mn} + \frac{1}{2} L_{ijkl} A^{ij} \wedge dA^{kl} \ .
\end{aligned} \tag{3.15}$$

These expressions are valid only up to total derivative terms that will be discarded.

Clearly there is a certain degree of factorisation for the Bagger-Lambert lagrangian into separate terms living on the different components of $\bigoplus_{\alpha=1}^{N} W_\alpha \oplus \bigoplus_{\pi=1}^{M} E_\pi \oplus E_0$. Indeed let us define accordingly $\mathscr{L}^W = -\frac{1}{2}\sum_{\alpha=1}^{N} D_\mu X_I^{a_\alpha} D^\mu X_I^{a_\alpha} + \mathscr{V}^W(X) + \mathscr{L}_{\rm CS}^W$ and likewise for E and E_0. This is mainly for notational convenience however and one must be wary of the fact that $\mathscr{L}^E$ and $\mathscr{L}^{E_0}$ could have some fields, namely components of A^{ij}, in common.

To relate the full lagrangian $\mathscr{L}$ with a super Yang-Mills theory, one has first to identify and integrate out those fields which are auxiliary or appear linearly as Lagrange multipliers. This will be most easily done by considering $\mathscr{L}^W$, $\mathscr{L}^E$ and $\mathscr{L}^{E_0}$ in turn.

3.2.1 $\mathscr{L}^W$

The field B^{a_α} appears only algebraically as an auxiliary field in $\mathscr{L}^W$. Its equation of motion implies

$$2\, Y^{\kappa^\alpha \kappa^\alpha} B^{a_\alpha} = X_I^{\kappa^\alpha} \mathscr{D} X_I^{a_\alpha} + *\mathscr{F}^{a_\alpha} \ , \tag{3.16}$$

for each value of α. Substituting this back into $\mathscr{L}^W$ then gives

$$-\frac{1}{2}\sum_{\alpha=1}^{N} D_\mu X_I^{a_\alpha} D^\mu X_I^{a_\alpha} + \mathscr{L}_{\rm CS}^W = \sum_{\alpha=1}^{N} \left\{ -\frac{1}{2} P_{IJ}^{\kappa^\alpha} \mathscr{D}_\mu X_I^{a_\alpha} \mathscr{D}^\mu X_J^{a_\alpha} - \frac{1}{4 Y^{\kappa^\alpha \kappa^\alpha}} \mathscr{F}_{\mu\nu}^{a_\alpha} \mathscr{F}^{\mu\nu\, a_\alpha} \right\} , \tag{3.17}$$

where, for each α, $P_{IJ}^{\kappa^\alpha} := \delta_{IJ} - \frac{X_I^{\kappa^\alpha} X_J^{\kappa^\alpha}}{Y^{\kappa^\alpha \kappa^\alpha}}$ is the projection operator onto the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ which is orthogonal to the 8-vector $X_I^{\kappa^\alpha}$ that κ_i^α projects the constant X_I^i onto.

Furthermore, in terms of the Lie bracket $[-,-]_\alpha$ on $\mathfrak{g}_\alpha$, the scalar potential can be written

$$\mathscr{V}^W(X) = -\frac{1}{4}\sum_{\alpha=1}^{N} Y^{\kappa^\alpha\kappa^\alpha}\, P^{\kappa^\alpha}_{IK}P^{\kappa^\alpha}_{JL}\,[X_I,X_J]^{a_\alpha}_\alpha [X_K,X_L]^{a_\alpha}_\alpha \ . \tag{3.18}$$

In conclusion, we have shown that upon integrating out B^{a_α} one can identify

$$\mathscr{L}^W = \sum_{\alpha=1}^{N} \mathscr{L}^{\rm SYM}\left(\mathscr{A}^{a_\alpha}, P^{\kappa^\alpha}_{IJ}X^{a_\alpha}_J, \|X^{\kappa^\alpha}\| |\mathfrak{g}_\alpha\right) \ . \tag{3.19}$$

The identification above with the lagrangian in (3.4) has revealed a rather intricate relation between the data κ^α_i and $\mathfrak{g}_\alpha$ on W_α from theorem 2 and the physical parameters in the super Yang-Mills theory. In particular, the coupling constant for the super Yang-Mills theory on W_α corresponds to the SO(8)-norm of $X^{\kappa^\alpha}_I$. Moreover, the direction of $X^{\kappa^\alpha}_I$ in $\mathbb{R}^8$ determines which hyperplane the seven scalar fields in the super Yang-Mills theory must occupy and thus may be different on each W_α. The gauge symmetry is based on the euclidean Lie algebra $\bigoplus_{\alpha=1}^N \mathfrak{g}_\alpha$.

The main point to emphasise is that it is the projections of the individual κ^α_i onto the vacuum described by constant X^i_I (rather than the vacuum expectation values themselves) which determine the physical moduli in the theory. For example, take $N=1$ with only one simple Lie algebra structure $\mathfrak{g} = \mathfrak{su}(n)$ on W. The lagrangian (3.19) then describes precisely the low-energy effective theory for n coincident D2-branes in type IIA string theory, irrespective of the index r of the initial 3-Lie algebra. The only difference is that the coupling $\|X^\kappa\|$, to be interpreted as the perimeter of the M-theory circle, is realised as a different projection for different values of r.

Thus, in general, we are assuming a suitably generic situation wherein none of the projections $X^{\kappa^\alpha}_I$ vanish identically. If $X^{\kappa^\alpha}_I = 0$ for a given value of α then the W_α part of the scalar potential (3.14) vanishes identically and the only occurrence of the corresponding B^{a_α} is in the Chern-Simons term (3.15). Thus, for this particular value of α, B^{a_α} has become a Lagrange multiplier imposing $\mathscr{F}^{a_\alpha}=0$ and so $\mathscr{A}^{a_\alpha}$ is pure gauge. The resulting lagrangian on this W_α therefore describes a free $N=8$ supersymmetric theory for the eight scalar fields $X^{a_\alpha}_I$.

3.2.2 $\mathscr{L}^E$

The field $\epsilon^{a_\pi b_\pi}A^{a_\pi b_\pi}$ appears only linearly in one term in $\mathscr{L}^E_{\rm CS}$ and is therefore a Lagrange multiplier imposing the constraint $A^{\eta^\pi\zeta^\pi} = d\phi^{\eta^\pi\zeta^\pi}$, for some some scalar fields $\phi^{\eta^\pi\zeta^\pi}$, for each value of π. The number of distinct scalars $\phi^{\eta^\pi\zeta^\pi}$ will depend on the number of linearly independent 2-planes in $\mathbb{R}^r$ which the collection of all $\eta^\pi\wedge\zeta^\pi$ span for $\pi = 1,\ldots,M$. Let us henceforth call this number k, which is clearly bounded above by $\binom{r}{2}$.

Moreover, up to total derivatives, one has a choice of taking just one of the two gauge fields $A^{\eta^\pi a_\pi}$ and $A^{\zeta^\pi a_\pi}$ to be auxiliary in $\mathscr{L}^E$. These are linearly independent gauge fields by virtue of the fact that $\eta^\pi\wedge\zeta^\pi$ span a 2-plane in $\mathbb{R}^r$ for each value of π. Without loss of generality we can take $A^{\eta^\pi a_\pi}$ to be auxiliary and integrate it out in favour of $A^{\zeta^\pi a_\pi}$.

After implementing the Lagrange multiplier constraint above, one finds that the equation of motion of $A^{\eta^\pi a_\pi}$ implies

$$2\,Y^{\zeta^\pi\zeta^\pi} A^{\eta^\pi a_\pi} = -\epsilon^{a_\pi b_\pi} \Big\{ X_I^{\zeta^\pi} \Big(dX_I^{b_\pi} + 2\,\epsilon^{b_\pi c_\pi} \Big(X_I^{c_\pi} d\phi^{\eta^\pi\zeta^\pi} + X_I^{\eta^\pi} A^{\zeta^\pi c_\pi} \Big)\Big) + 2 * \Big(dA^{\zeta^\pi b_\pi} + 2\,\epsilon^{b_\pi c_\pi} d\phi^{\eta^\pi\zeta^\pi} \wedge A^{\zeta^\pi c_\pi} \Big) \Big\} \ . \tag{3.20}$$

Substituting this back into $\mathscr{L}^E$ then, following a rather lengthy but straightforward calculation, one finds that

$$\begin{aligned} -\frac{1}{2}\sum_{\pi=1}^{M} D_\mu X_I^{a_\pi} D^\mu X_I^{a_\pi} + \mathscr{L}^E_{\mathrm{CS}} = &-\frac{1}{2}\sum_{\pi=1}^{M} P_{IJ}^{\zeta^\pi} \Big(\partial_\mu X_I^{a_\pi} + 2\,\epsilon^{a_\pi b_\pi} \Big(X_I^{b_\pi} \partial_\mu \phi^{\eta^\pi\zeta^\pi} + X_I^{\eta^\pi} A_\mu^{\zeta^\pi b_\pi} \Big)\Big) \\ &\times \Big(\partial^\mu X_J^{a_\pi} + 2\,\epsilon^{a_\pi c_\pi} \Big(X_J^{c_\pi} \partial^\mu \phi^{\eta^\pi\zeta^\pi} + X_J^{\eta^\pi} A^{\mu\,\zeta^\pi c_\pi} \Big)\Big) \\ &- \sum_{\pi=1}^{M} \frac{4}{Y^{\zeta^\pi\zeta^\pi}} \Big(\partial_{[\mu} A_{\nu]}^{\zeta^\pi a_\pi} + 2\,\epsilon^{a_\pi b_\pi} \partial_{[\mu} \phi^{\eta^\pi\zeta^\pi} A_{\nu]}^{\zeta^\pi b_\pi} \Big) \\ &\times \Big(\partial^\mu A^{\nu\,\zeta^\pi a_\pi} + 2\,\epsilon^{a_\pi c_\pi} \partial^\mu \phi^{\eta^\pi\zeta^\pi} A^{\nu\,\zeta^\pi c_\pi} \Big) \ , \end{aligned} \tag{3.21}$$

where, for each π, $P_{IJ}^{\zeta^\pi} := \delta_{IJ} - \frac{X_I^{\zeta^\pi} X_J^{\zeta^\pi}}{Y^{\zeta^\pi\zeta^\pi}}$ projects onto the hyperplane $\mathbb{R}^7 \subset \mathbb{R}^8$ orthogonal to the 8-vector $X_I^{\zeta^\pi}$ which ζ_i^π projects the constant X_I^i onto.

We have deliberately written (3.21) in a way that is suggestive of a super Yang-Mills description for the fields on E however, in contrast with the preceding analysis for W, the gauge structure here is not quite so manifest. To make it more transparent, let us fix a particular value of π and consider a 4-dimensional lorentzian vector space of the form $\mathbb{R}e_+ \oplus \mathbb{R}e_- \oplus E_\pi$, where the particular basis (e_+, e_-) for the two null directions obeying $\langle e_+, e_- \rangle = 1$ and $\langle e_\pm, e_\pm \rangle = 0 = \langle e_\pm, e_{a_\pi} \rangle$ can of course depend on the choice of π (we will omit the π label here though). If we take E_π to be a euclidean 2-dimensional abelian Lie algebra then we can define a lorentzian metric Lie algebra structure on $\mathbb{R}e_+ \oplus \mathbb{R}e_- \oplus E_\pi$ given by the double extension $\mathfrak{d}(E_\pi, \mathbb{R})$. The nonvanishing Lie brackets of $\mathfrak{d}(E_\pi, \mathbb{R})$ are

$$[e_+, e_{a_\pi}] = -\epsilon_{a_\pi b_\pi} e_{b_\pi} \ , \qquad [e_{a_\pi}, e_{b_\pi}] = -\epsilon_{a_\pi b_\pi} e_- \ . \tag{3.22}$$

This double extension is precisely the Nappi-Witten Lie algebra.

For each value of π we can collect the following sets of scalars $\mathsf{X}_I^\pi := (X_I^{\eta^\pi}, X_I^{\zeta^\pi}, X_I^{a_\pi})$ and gauge fields $\mathsf{A}^\pi := (2\,d\phi^{\eta^\pi\zeta^\pi}, 0, -2\,A^{\zeta^\pi a_\pi})$ into elements of the aforementioned vector space $\mathbb{R}e_+ \oplus \mathbb{R}e_- \oplus E_\pi$. The virtue of doing so being that if $\mathsf{D} = d + [\mathsf{A}, -]$, for each value of π, is the canonical gauge-covariant derivative with respect to each $\mathfrak{d}(E_\pi, \mathbb{R})$ then $(\mathsf{D}\mathsf{X}_I)^{a_\pi} = dX_I^{a_\pi} + 2\,\epsilon^{a_\pi b_\pi} \Big(X_I^{b_\pi} d\phi^{\eta^\pi\zeta^\pi} + X_I^{\eta^\pi} A^{\zeta^\pi b_\pi} \Big)$ while the associated field strength $\mathsf{F}_{\mu\nu} = [\mathsf{D}_\mu, \mathsf{D}_\nu]$ has $\mathsf{F}^{a_\pi} = -2 \left(dA^{\zeta^\pi a_\pi} + 2\,\epsilon^{a_\pi b_\pi} d\phi^{\eta^\pi\zeta^\pi} \wedge A^{\zeta^\pi b_\pi} \right)$. These are exactly the components appearing in (3.21)!

Moreover, the scalar potential $\mathscr{V}^E(X)$ can be written

$$\mathscr{V}^E(X) = -\frac{1}{4}\sum_{\pi=1}^{M} Y^{\zeta^\pi\zeta^\pi} P_{IK}^{\zeta^\pi} P_{JL}^{\zeta^\pi} [\mathsf{X}_I, \mathsf{X}_J]^{a_\pi} [\mathsf{X}_K, \mathsf{X}_L]^{a_\pi} \ , \tag{3.23}$$

where $[-,-]$ denotes the Lie bracket on each $\mathfrak{d}(E_\pi,\mathbb{R})$ factor.

Thus it might appear that $\mathscr{L}^E$ is going to describe a super Yang-Mills theory whose gauge algebra is $\bigoplus_{\pi=1}^M \mathfrak{d}(E_\pi,\mathbb{R})$, which indeed has a maximally isotropic centre and so is of the form noted in section 3.1.2. However, this need not be the case in general since the functions $\phi^{\eta^\pi\zeta^\pi}$ appearing in the e_+ direction of each A^π must describe the same degree of freedom for different values of π precisely when the corresponding 2-planes in $\mathbb{R}^r$ spanned by $\eta^\pi\wedge\zeta^\pi$ are linearly dependent. Consequently we must identify the (e_+,e_-) directions in all those factors $\mathfrak{d}(E_\pi,\mathbb{R})$ for which the associated $\eta^\pi\wedge\zeta^\pi$ span the same 2-plane in $\mathbb{R}^r$. It is not hard to see that, with respect to a general basis on $\bigoplus_{\pi=1}^M E_\pi$, the resulting Lie algebra $\mathfrak{k}$ must take the form $\bigoplus_{[\pi]=1}^k \mathfrak{d}(E_{[\pi]},\mathbb{R})$ of an orthogonal direct sum over the number of independent 2-planes k spanned by $\eta^{[\pi]}\wedge\zeta^{[\pi]}$ of a set of k double extensions $\mathfrak{d}(E_{[\pi]},\mathbb{R})$ of even-dimensional vector spaces $E_{[\pi]}$, where $\bigoplus_{\pi=1}^M E_\pi = \bigoplus_{[\pi]=1}^k E_{[\pi]}$. That is each $[\pi]$ can be thought of as encompassing an equivalence class of π values for which the corresponding 2-forms $\eta^\pi\wedge\zeta^\pi$ are all proportional to each other. The data for $\mathfrak{k}$ therefore corresponds to a set of k nondegenerate elements $J_{[\pi]}\in\mathfrak{so}(E_{[\pi]})$ where, for a given value of $[\pi]$, the relative eigenvalues of $J_{[\pi]}$ are precisely the relative proportionality constants for the linearly dependent 2-forms $\eta^\pi\wedge\zeta^\pi$ in the equivalence class. Clearly $\mathfrak{k}$ therefore has index k, dimension $2\left(k+\left[\frac{\dim W_0}{2}\right]\right)$ and admits a maximally isotropic centre.

Putting all this together, we conclude that

$$\mathscr{L}^E = \sum_{[\pi]=1}^{k} \mathscr{L}^{\mathrm{SYM}}\left(\mathsf{A}^{[\pi]}, P_{IJ}^{\zeta^{[\pi]}}\mathsf{X}_J^{[\pi]}, \|X^{\zeta^{[\pi]}}\| \middle| \mathfrak{d}(E_{[\pi]},\mathbb{R})\right) . \tag{3.24}$$

One can check from (3.14) and (3.21) that the contributions to the Bagger-Lambert lagrangian on E coming from different E_π factors, but with π values in the same equivalence class $[\pi]$, are precisely accounted for in the expression (3.24) by the definition above of the elements $J_{[\pi]}$ defining the double extensions.

The identification above again provides quite an intricate relation between the data on E_π from theorem 2 and the physical super Yang-Mills parameters. However, we know from section 3.1.2 that the physical content of super Yang-Mills theories whose gauge symmetry is based on a lorentzian Lie algebra corresponding to a double extension is rather more simple, being described in terms of free massive vector supermultiplets. Let us therefore apply this preceding analysis to the theory above.

The description above of the lagrangian on each factor E_π has involved projecting degrees of freedom onto the hyperplane $\mathbb{R}^7\subset\mathbb{R}^8$ orthogonal to $X_I^{\zeta^\pi}$. The natural analogy here of the six-dimensional subspace occupied by the massive scalar fields in section 3.1.2 is obtained by projecting onto the subspace $\mathbb{R}^6\subset\mathbb{R}^8$ which is orthogonal to the plane in $\mathbb{R}^8$ spanned by $X^{\eta^\pi}\wedge X^{\zeta^\pi}$, i.e. the image in $\Lambda^2\mathbb{R}^8$ of the 2-form $\eta^\pi\wedge\zeta^\pi$ under the map from $\mathbb{R}^r\to\mathbb{R}^8$ provided by the vacuum expectation values X_I^i. This projection operator can be written

$$P_{IJ}^{\eta^\pi\zeta^\pi} = \delta_{IJ} - X_I^{\eta^\pi}Q_J^{\eta^\pi} - X_I^{\zeta^\pi}Q_J^{\zeta^\pi} , \tag{3.25}$$

where

$$Q_I^{\eta^\pi} := \frac{1}{(\Delta_{\eta^\pi\zeta^\pi})^2}\left(Y^{\zeta^\pi\zeta^\pi} X_I^{\eta^\pi} - Y^{\eta^\pi\zeta^\pi} X_I^{\zeta^\pi}\right)$$
$$Q_I^{\zeta^\pi} := \frac{1}{(\Delta_{\eta^\pi\zeta^\pi})^2}\left(Y^{\eta^\pi\eta^\pi} X_I^{\zeta^\pi} - Y^{\eta^\pi\zeta^\pi} X_I^{\eta^\pi}\right) , \tag{3.26}$$

and

$$(\Delta_{\eta^\pi\zeta^\pi})^2 := \| X^{\eta^\pi} \wedge X^{\zeta^\pi} \|^2 \equiv Y^{\eta^\pi\eta^\pi} Y^{\zeta^\pi\zeta^\pi} - (Y^{\eta^\pi\zeta^\pi})^2 . \tag{3.27}$$

The quantities defined in (3.26) are the dual elements to $X_I^{\eta^\pi}$ and $X_I^{\zeta^\pi}$ such that $Q_I^{\eta^\pi} X_I^{\eta^\pi} = 1 = Q_I^{\zeta^\pi} X_I^{\zeta^\pi}$ and $Q_I^{\eta^\pi} X_I^{\zeta^\pi} = 0 = Q_I^{\zeta^\pi} X_I^{\eta^\pi}$. The expression (3.27) identifies $\Delta_{\eta^\pi\zeta^\pi}$ with the area in $\mathbb{R}^8$ spanned by $X^{\eta^\pi} \wedge X^{\zeta^\pi}$. From these definitions, it follows that $P_{IJ}^{\eta^\pi\zeta^\pi}$ in (3.25) indeed obeys $P_{IJ}^{\eta^\pi\zeta^\pi} = P_{JI}^{\eta^\pi\zeta^\pi}$, $P_{IK}^{\eta^\pi\zeta^\pi} P_{JK}^{\eta^\pi\zeta^\pi} = P_{IJ}^{\eta^\pi\zeta^\pi}$ and $P_{IJ}^{\eta^\pi\zeta^\pi} X_J^{\eta^\pi} = 0 = P_{IJ}^{\eta^\pi\zeta^\pi} X_J^{\zeta^\pi}$.

The scalar potential (3.23) on E has a natural expression in terms of the objects defined in (3.25) and (3.27) as

$$\mathscr{V}^E(X) = -\frac{1}{2}\sum_{\pi=1}^{M} (\Delta_{\eta^\pi\zeta^\pi})^2 \, P_{IJ}^{\eta^\pi\zeta^\pi} X_I^{a_\pi} X_J^{a_\pi} . \tag{3.28}$$

Furthermore, using the identity

$$P_{IJ}^{\eta^\pi\zeta^\pi} \equiv P_{IJ}^{\zeta^\pi} - \frac{(\Delta_{\eta^\pi\zeta^\pi})^2}{Y^{\zeta^\pi\zeta^\pi}} Q_I^{\eta^\pi} Q_J^{\eta^\pi} , \tag{3.29}$$

allows one to reexpress the remaining terms

$$-\frac{1}{2}\sum_{\pi=1}^{M} D_\mu X_I^{a_\pi} D^\mu X_I^{a_\pi} + \mathscr{L}_{\mathrm{CS}}^E \tag{3.30}$$

in (3.21) as

$$\begin{aligned}\sum_{\pi=1}^{M} -\frac{1}{2} P_{IJ}^{\eta^\pi\zeta^\pi} \mathcal{D}_\mu X_I^{a_\pi} \mathcal{D}^\mu X_J^{a_\pi} &- \frac{1}{Y^{\zeta^\pi\zeta^\pi}}\left(2\,\mathcal{D}_{[\mu} A_{\nu]}^{\zeta^\pi a_\pi}\right)\left(2\,\mathcal{D}^\mu A^{\nu\,\zeta^\pi a_\pi}\right) \\ -\frac{1}{2}\sum_{\pi=1}^{M} \frac{Y^{\zeta^\pi\zeta^\pi}}{(\Delta_{\eta^\pi\zeta^\pi})^2} &\left(X_I^{\eta^\pi} P_{IJ}^{\zeta^\pi} \mathcal{D}_\mu X_J^{a_\pi} + 2\,\frac{(\Delta_{\eta^\pi\zeta^\pi})^2}{Y^{\zeta^\pi\zeta^\pi}} \epsilon^{a_\pi b_\pi} A_\mu^{\zeta^\pi b_\pi}\right) \\ &\times \left(X_K^{\eta^\pi} P_{KL}^{\zeta^\pi} \mathcal{D}^\mu X_L^{a_\pi} + 2\,\frac{(\Delta_{\eta^\pi\zeta^\pi})^2}{Y^{\zeta^\pi\zeta^\pi}} \epsilon^{a_\pi c_\pi} A^{\mu\,\zeta^\pi c_\pi}\right) ,\end{aligned} \tag{3.31}$$

where we have introduced the covariant derivative $\mathcal{D}\Phi^{a_\pi} := d\Phi^{a_\pi} + 2\,\epsilon^{a_\pi b_\pi}\, d\phi^{\eta^\pi\zeta^\pi} \wedge \Phi^{b_\pi}$ for any differential form Φ^{a_π} on $\mathbb{R}^{1,2}$ taking values in E_π. Similar to what we saw in section 3.1.2, the six projected scalars $P_{IJ}^{\eta^\pi\zeta^\pi} X_J^{a_\pi}$ in the first line of (3.31) do not couple to the gauge field $A^{\zeta^\pi a_\pi}$ on each E_π. Moreover, the remaining scalar in the second line of (3.31) can be eliminated from the lagrangian, for each E_π, using the gauge symmetry under which $\delta A^{i a_\pi} = \mathcal{D}\Lambda^{i a_\pi}$ for any parameter $\Lambda^{i a_\pi}$ to fix $\Lambda^{\zeta^\pi a_\pi} = -\frac{1}{2}\frac{Y^{\zeta^\pi\zeta^\pi}}{(\Delta_{\eta^\pi\zeta^\pi})^2}\epsilon^{a_\pi b_\pi} X_I^{\eta^\pi} P_{IJ}^{\zeta^\pi} X_J^{b_\pi}$. There is a remaining gauge symmetry under which $\delta\phi^{\eta^\pi\zeta^\pi} = \Lambda^{\eta^\pi\zeta^\pi}$ and $\delta\Phi^{a_\pi} = -2\,\Lambda^{\eta^\pi\zeta^\pi}\epsilon^{a_\pi b_\pi}\Phi^{b_\pi}$ where the gauge parameter $\Lambda^{\eta^\pi\zeta^\pi} = \eta_i^\pi \zeta_j^\pi \Lambda^{ij}$, under which the derivative $\mathcal{D}$ transforms covariantly. This can also be fixed to set $\mathcal{D} = d$ on each E_π. Notice that one has precisely the right number of these gauge symmetries to fix all the independent projections $\phi^{\eta^\pi \zeta^\pi}$ appearing in the covariant derivatives.

After doing this one combines (3.28) and (3.31) to write

$$\begin{aligned} \mathscr{L}^E = & \sum_{\pi=1}^{M} -\frac{1}{2} P_{IJ}^{\eta^\pi \zeta^\pi} \partial_\mu X_I^{a_\pi} \partial^\mu X_J^{a_\pi} - \frac{1}{2} (\Delta_{\eta^\pi \zeta^\pi})^2 P_{IJ}^{\eta^\pi \zeta^\pi} X_I^{a_\pi} X_J^{a_\pi} \\ & + \sum_{\pi=1}^{M} -\frac{1}{Y^{\zeta^\pi \zeta^\pi}} (2\, \partial_{[\mu} A_{\nu]}^{\zeta^\pi a_\pi}) (2\, \partial^{[\mu} A^{\nu]\, \zeta^\pi a_\pi}) - \frac{2}{Y^{\zeta^\pi \zeta^\pi}} (\Delta_{\eta^\pi \zeta^\pi})^2 A_\mu^{\zeta^\pi a_\pi} A^{\mu\, \zeta^\pi a_\pi}\,, \end{aligned} \tag{3.32}$$

describing precisely the bosonic part of the lagrangian for free decoupled abelian $N = 8$ massive vector supermultiplets on each E_π, whose bosonic fields comprise the six scalars $P_{IJ}^{\eta^\pi \zeta^\pi} X_J^{a_\pi}$ and gauge field $-2\frac{1}{\|X^{\zeta^\pi}\|} A^{\zeta^\pi a_\pi}$, all with mass $\Delta_{\eta^\pi \zeta^\pi}$ on each E_π. It is worth stressing that we have presented (3.32) as a sum over all E_π just so that the masses $\Delta_{\eta^\pi \zeta^\pi}$ on each factor can be written more explicitly. We could equally well have presented things in terms of a sum over the equivalence classes $E_{[\pi]}$, as in (3.24), whereby the relative proportionality constants for the $\Delta_{\eta^\pi \zeta^\pi}$ within a given class $[\pi]$ would be absorbed into the definition of the corresponding $J_{[\pi]}$.

The lagrangian on a given E_π in the sum (3.32) can also be obtained from the truncation of an $N = 8$ super Yang-Mills theory with euclidean gauge algebra $\mathfrak{g}$ via the procedure described at the end of section 3.1.2. In particular, let us identify a given E_π with the Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ of rank two. Then we require $-\|X^{\zeta^\pi}\|^2 (\mathrm{ad}_y)^2 = (\Delta_{\eta^\pi \zeta^\pi})^2 \mathbf{1}_2$ on E_π for some constant $y \in E_\pi^\perp \subset \mathfrak{g}$. In this case $\mathfrak{g}$ must be either $\mathfrak{su}(3)$, $\mathfrak{so}(5)$, $\mathfrak{so}(4)$ or $\mathfrak{g}_2$ and $E_\pi^\perp$ is identified with the root space of $\mathfrak{g}$ whose dimension is 6, 8, 4 or 12 respectively. A solution in this case is to take y proportional to the vector with only +1/-1 entries along the positive/negative roots of $\mathfrak{g}$. The proportionality constant here being $\frac{\Delta_{\eta^\pi \zeta^\pi}}{\sqrt{h(\mathfrak{g})}\|X^{\zeta^\pi}\|}$ where $h(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$ and equals 3, 3, 2 or 4 for $\mathfrak{su}(3)$, $\mathfrak{so}(5)$, $\mathfrak{so}(4)$ or $\mathfrak{g}_2$ respectively (it is assumed that the longest root has norm-squared equal to 2 with respect to the Killing form in each case).

Recall from [23] that several of these rank two Lie algebras are thought to correspond to the gauge algebras for $N = 8$ super Yang-Mills theories whose IR superconformal fixed points are described by the Bagger-Lambert theory based on S_4 for two M2-branes on $\mathbb{R}^8/\mathbb{Z}_2$ (with Lie algebras $\mathfrak{so}(4)$, $\mathfrak{so}(5)$ and $\mathfrak{g}_2$ corresponding to Chern-Simons levels $k = 1, 2, 3$). It would interesting to understand whether there is any relation with the aforementioned truncation beyond just numerology! The general mass formulae we have obtained are somewhat reminiscent of equation (26) in [23] for the BLG model based on S_4 which describes the mass in terms of the area of the triangle formed between the location of the two M2-branes and the orbifold fixed point on $\mathbb{R}^8/\mathbb{Z}_2$. More generally, it would be interesting to understand whether there is a specific D-brane configuration for which $\mathscr{L}^E$ is the low-energy effective lagrangian?

3.2.3 $\mathscr{L}^{E_0}$

The field A^{ia_0} appears only linearly in one term in $\mathscr{L}^0_{\rm CS}$ and is therefore a Lagrange multiplier imposing the constraint $K_{ijka_0}A^{jk} = d\gamma_{ia_0}$, where γ_{ia_0} is a scalar field on $\mathbb{R}^{1,2}$ taking values in $\mathbb{R}^r \otimes E_0$.

Substituting this condition into the lagrangian allows us to write

$$\begin{aligned} -\frac{1}{2}D_\mu X_I^{a_0} D^\mu X_I^{a_0} + \mathscr{L}^{E_0}_{\rm CS} = &-\frac{1}{2}\partial_\mu \left(X_I^{a_0} - \gamma_i{}^{a_0} X_I^i \right) \partial^\mu \left(X_I^{a_0} - \gamma_j{}^{a_0} X_I^j \right) \\ &- \frac{1}{3} A^{ij} \wedge d\gamma_{ia_0} \wedge d\gamma_{ja_0} + \frac{1}{2} L_{ijkl} A^{ij} \wedge dA^{kl} \, . \end{aligned} \tag{3.33}$$

The first line shows that we can simply redefine the scalars $X_I^{a_0}$ such that they decouple and do not interact with any other fields in the theory.

Notice that none of the projections $A^{\eta^\pi \zeta^\pi} = d\phi^{\eta^\pi \zeta^\pi}$ of A^{ij} that appeared in $\mathscr{L}^E$ can appear in the second line of (3.33) since the corresponding terms would be total derivatives. Consequently, our indifference to $\mathscr{L}^{E_0}$ in the gauge-fixing that was described for $\mathscr{L}^E$, resulting in (3.32), was indeed legitimate. Furthermore, there can be no components of A^{ij} along the 2-planes in $\mathbb{R}^r$ spanned by the nonanishing components of K_{ijka_0} here for the same reason.

The contribution coming from the Chern-Simons term in the second line of (3.33) is therefore completely decoupled from all the other terms in the lagrangian. It has a rather unusual-looking residual gauge symmetry, inherited from that in the original Bagger-Lambert theory, under which $\delta\gamma_{ia_0} = \sigma_{ia_0} := K_{ia_0kl}\Lambda^{kl}$ and $L_{ijkl}\left(\delta A^{kl} - d\Lambda^{kl}\right) = \sigma_{[i}{}^{a_0} d\gamma_{j]a_0}$ for any gauge parameter Λ^{ij}. In addition to the second line of (3.33) being invariant under this gauge transformation, one can easily check that so is the tensor $L_{ijkl}dA^{kl} - d\gamma_{ia_0} \wedge d\gamma_{ja_0}$. This is perhaps not surprising since the vanishing of this tensor is precisely the field equation resulting from varying A^{ij} in the second line of (3.33). The important point though is that this gauge-invariant tensor is exact and thus the field equations resulting from the second line of (3.33) are precisely equivalent to those obtained from an abelian Chern-Simons term for the gauge field $C_{ij} := L_{ijkl}A^{kl} - \gamma_{[i}{}^{a_0} \wedge d\gamma_{j]a_0}$ (where the $[ij]$ indices do not run over any 2-planes in $\mathbb{R}^r$ which are spanned by the nonvanishing components of $\eta^\pi_{[i}\zeta^\pi_{j]}$ and K_{ijka_0}).

In summary, up to the aforementioned field redefinitions, we have found that

$$\mathscr{L}^{E_0} = -\frac{1}{2}\partial_\mu X_I^{a_0} \partial^\mu X_I^{a_0} + \frac{1}{2} M^{ijkl} C_{ij} \wedge dC_{kl} \, , \tag{3.34}$$

for some constant tensor M^{ijkl}, which can be taken to obey $M^{ijkl} = M^{[ij][kl]} = M^{klij}$, that is generically a complicated function of the components L_{ijkl} and K_{ijka_0}. Clearly this redefined abelian Chern-Simons term is only well-defined in a path integral provided the components M^{ijkl} are quantised in suitable integer units. However, since none of the dynamical fields are charged under C_{ij} then we conclude that the contribution from $\mathscr{L}^{E_0}$ is essentially trivial.

3.3 Examples

Let us end by briefly describing an application of this formalism to describe the unitary gauge theory resulting from the Bagger-Lambert theory associated with two of the admissible index-2 3-Lie algebras in the IIIb family from [13] that were detailed in section 2.3.

3.3.1 $V_{\mathbf{IIIb}}(0,0,0,\mathfrak{h},\mathfrak{g},\psi)$

The data needed for this in theorem 2 is $\kappa|_{\mathfrak{h}} = (0,1)^t$, $\kappa|_{\mathfrak{g}_\alpha} = (\psi_\alpha, 1)^t$. The resulting Bagger-Lambert lagrangian will only get a contribution from $\mathscr{L}^W$ and describes a sum of separate $N = 8$ super Yang-Mills lagrangians on $\mathfrak{h}$ and on each factor $\mathfrak{g}_\alpha$, with the respective euclidean Lie algebra structures describing the gauge symmetry. The super Yang-Mills theory on $\mathfrak{h}$ has coupling $\|X^{u_2}\|$ and the seven scalar fields occupy the hyperplane orthogonal to X^{u_2} in $\mathbb{R}^8$. Similarly, the $N = 8$ theory on a given $\mathfrak{g}_\alpha$ has coupling $\|\psi_\alpha X^{u_1} + X^{u_2}\|$ with scalars in the hyperplane orthogonal to $\psi_\alpha X^{u_1} + X^{u_2}$. This is again generically a super Yang-Mills theory though it degenerates to a maximally supersymmetric free theory for all eight scalars if there are any values of α for which $\psi_\alpha X^{u_1} + X^{u_2} = 0$.

3.3.2 $V_{\mathbf{IIIb}}(E,J,0,\mathfrak{h},0,0)$

The data needed for this in theorem 2 is $\kappa|_{\mathfrak{h}} = (0,1)^t$ and $J^\pi = \eta^\pi \wedge \zeta^\pi$ where η^π and ζ^π are 2-vectors spanning $\mathbb{R}^2$ for each value of π and $E = \bigoplus_{\pi=1}^M E_\pi$. The data comprising J^π can also be understood as a special case of a general admissible index r 3-Lie algebra having all $\eta^\pi \wedge \zeta^\pi$ spanning the same 2-plane in $\mathbb{R}^r$ (when $r = 2$ this is unavoidable, of course). The resulting Bagger-Lambert lagrangian will get one contribution from $\mathscr{L}^W$, describing precisely the same $N = 8$ super Yang-Mills theory on $\mathfrak{h}$ we saw above, and one contribution from $\mathscr{L}^E$. The latter being the simplest case of the lagrangian (3.24) where there is just one equivalence class of 2-planes spanned by all $\eta^\pi \wedge \zeta^\pi$ and the gauge symmetry is based on the lorentzian Lie algebra $\mathfrak{d}(E,\mathbb{R})$. The physical degrees of freedom describe free abelian $N = 8$ massive vector supermultiplets on each E_π with masses $\Delta_{\eta^\pi\zeta^\pi}$ as in (3.32). Mutatis mutandis, this example is equivalent to the Bagger-Lambert theory resulting from the most general finite-dimensional 3-Lie algebra example considered in section 4.3 of [18].

Acknowledgments

EME would like to thank Iain Gordon for useful discussions. PdM is supported by a Seggie-Brown Postdoctoral Fellowship of the School of Mathematics of the University of Edinburgh.

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